ISSN:2636-7467(Online)

international journal of MAPS IN MATHEMATICS

Volume 1
Issue 1
2018

Editor in chief

Bayram SAHIN
Ege University, Izmir, TURKEY

International Journal of Maps in Mathematics is a fully refereed journal devoted to publishing recent results obtained in the research areas of maps in mathematics

www.journalmim.com

VOLUME 1 ISSUE 1
ISSN 2636-7467

March 2018
https://www.journalmim.com

International Journal of Maps in Mathematics

International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:1-14

www.journalmim.com

SOME RESULTS ON COMMON FIXED POINTS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN COMPLEX VALUED METRIC SPACE

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Abstract. In this manuscript, we have obtained the sufficient conditions for the existence and uniqueness of a pair of mappings satisfying rational type contractive conditions in the framework of complex valued metric space. Our result generalizes the well known result introduced by Azam et al. [2] in complex valued metric space. Also, various deductions have been provided.

1. Introduction

Azam et al. [2] introduced the concept of more general metric space, which is well known as complex valued metric spaces. He gave sufficient conditions for the existence and uniqueness of common fixed points satisfying contractive conditions. Later, S. Bhatt et al. [4] without using the notion of continuity proved a common fixed point theorem for weakly compatible maps in complex valued metric spaces. F. Rouzkard and M. Imdad [12] considering rational type contractive conditions proved some common fixed point theorems in the framework of complex valued metric space. C. Klin-eam and C. Suanoom [8] proved certain common fixed-point theorems for two single-valued mappings satisfy certain metric inequalities.

Received: 2018-02-16 *Accepted:2018-03-05*

2010 Mathematics Subject Classification: 47H10, 54H25.

Key words: fixed point theorem, contraction mapping, complex valued metric space

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The notion of complex valued metric space lead to development in non linear analysis. Thereafter, many results have been proved by the researchers in the framework of complex valued metric spaces for references (see [7]-[13]).

2. Preliminaries

To begin with, we recall some basic definitions, notations, and results. The following definitions of Azam et al. [2] are required in the sequel.

Let $\mathbb{C}$ be a set of complex number such that $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$, such that $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Img(z_1) \leq Img(z_2)$.
It follows that

$$z_1 \precsim z_2$$

if one of the below mentioned conditions is satisfied:

((i)) $Re(z_1) = Re(z_2)$, $Img(z_1) < Img(z_2)$;

((ii)) $Re(z_1) < Re(z_2)$, $Img(z_1) = Img(z_2)$;

((iii)) $Re(z_1) < Re(z_2)$, $Img(z_1) < Img(z_2)$;

((iv)) $Re(z_1) = Re(z_2)$, $Img(z_1) = Img(z_2)$.

In particular, we will write $z_1 \precnsim z_2$, if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied. We will write $z_1 \prec z_2$ if only (iii) is satisfied.

Remark 2.1. *We obtained that the following statements holds:*

- $a, b \in R$ *and* $a \leq b$ *implies* $az \precsim bz$, *for all* $z \in \mathbb{C}$*;*
- $0 \precsim z_1 \precnsim z_2$ *implies* $|z_1| < |z_2|$*;*
- $z_1 \preceq z_2$ *and* $z_2 \prec z_3$ *imply* $z_1 \prec z_3$.

Definition 2.1. [2] *Let X be non-empty set. Suppose that the mapping $\rho_c : X \times X \to \mathbb{C}$ satisfies the following conditions:*

((i)) $0 \precsim \rho_c(x, y)$ *for all* $x, y \in X$ *and* $\rho_c(x, y) = 0$ *if* $x = y$*;*

((ii)) $\rho_c(x, y) = \rho_c(y, x)$ *for all* $x, y \in X$*;*

((iii)) $\rho_c(x, y) \precsim \rho_c(x, z) + \rho_c(z, y)$ *for all* $x, y, z \in X$.

Then, ρ_c is called a complex valued metric on X, and (X, ρ_c) is called complex valued metric space.

Definition 2.2. [2] *A point $x \in X$ is called an interior of a set $A \subseteq X$ whenever their exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : \rho_c(x, y) \prec r\} \subseteq A$.*

Definition 2.3. [2] *Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$, there is $n_0 \in N$ such that for all $n > n_0$, $\rho_c(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in N$, such that for all $n > n_0$, $\rho_c(x_m, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, ρ_c).*

Definition 2.4. [2] *If every Cauchy sequence is convergent in (X, ρ_c) then (X, ρ_c) is called a complete complex valued metric space.*

Lemma 2.1. [2] *Let (X, ρ_c) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $|\rho_c(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2. [2] *Let (X, ρ_c) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\rho_c(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3. Some results on Fixed point

Theorem 3.1. *Let (X, ρ_c) be a complete complex valued metric space and $S, T : X \to X$ be self mappings satisfying the following condition:*

$$\rho_c(Sx, Ty) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma\frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y) + \rho_c(x, Ty) + \rho_c(y, Sx)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha + \beta + \gamma < 1$. Then S,T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point and define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$.

Then,

$$\begin{aligned}
\rho_c(x_{2k+1}, x_{2k+2}) &= \rho_c(Sx_{2k}, Tx_{2k+1}) \\
&\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1})} \\
&\quad + \gamma\frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, Tx_{2k+1}) + \rho_c(x_{2k+1}, Sx_{2k})} \\
&\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1})} \\
&\quad + \gamma\frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, x_{2k+2}) + \rho_c(x_{2k+1}, x_{2k+1})}
\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k},x_{2k+1}) &\leq 1+\rho_c(x_{2k},x_{2k+1}) \text{ and}\\ \rho_c(x_{2k},x_{2k+1}) &\leq 1+\rho_c(x_{2k},x_{2k+1})+\rho_c(x_{2k},x_{2k+2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+1},x_{2k+2}) &\precsim \alpha\rho_c(x_{2k},x_{2k+1})+\beta\rho_c(x_{2k+1},x_{2k+2})+\gamma\rho_c(x_{2k+1},x_{2k+2})\\ \rho_c(x_{2k+1},x_{2k+2}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k},x_{2k+1}).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_c(x_{2k+2},x_{2k+3}) &= \rho_c(x_{2k+3},x_{2k+2})=\rho_c(Sx_{2k+2},Tx_{2k+1})\\ &\precsim \alpha\rho_c(x_{2k+2},x_{2k+1})+\beta\frac{\rho_c(x_{2k+2},Sx_{2k+2})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k+2},x_{2k+1})}\\ &\quad+\gamma\frac{\rho_c(x_{2k+2},Sx_{2k+2})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k+2},x_{2k+1})+\rho_c(x_{2k+2},Tx_{2k+1})+\rho_c(x_{2k+1},Sx_{2k+2})}\\ &\precsim \alpha\rho_c(x_{2k+2},x_{2k+1})+\beta\frac{\rho_c(x_{2k+2},x_{2k+3})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k+2},x_{2k+1})}\\ &\quad+\gamma\frac{\rho_c(x_{2k+2},x_{2k+3})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k+2},x_{2k+1})+\rho_c(x_{2k+2},x_{2k+2})+\rho_c(x_{2k+1},x_{2k+3})}\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k+2},x_{2k+1}) &\leq 1+\rho_c(x_{2k+2},x_{2k+1}) \text{ and}\\ \rho_c(x_{2k+2},x_{2k+1}) &\leq 1+\rho_c(x_{2k+2},x_{2k+1})+\rho_c(x_{2k+1},x_{2k+3}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+2},x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+2},x_{2k+1})+\beta\rho_c(x_{2k+2},x_{2k+3})+\gamma\rho_c(x_{2k+2},x_{2k+3})\\ \rho_c(x_{2k+2},x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k+2},x_{2k+1})\\ &\text{or}\\ \rho_c(x_{2k+2},x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k+1},x_{2k+2}).\end{aligned}$$

Assume, $h=\dfrac{\alpha}{1-\beta-\gamma}<1$, we have

$$\rho_c(x_{n+1},x_{n+2})\precsim hd(x_n,x_{n+1})\precsim ...\precsim h^{n+1}\rho_c(x_0,x_1).$$

For some $m > n$, we have

$$\begin{aligned}
\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m) \\
&\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1) \\
&\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).
\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0, \ as \ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove that $\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$\begin{aligned}
\rho_c(u, Su) = z &\precsim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, Su) + \rho_c(u, Tx_{2k+1})} \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{zd(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2})}.
\end{aligned}$$

This implies,

$$\begin{aligned}
|\rho_c(u, Su)| &\leq |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\
&\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+1}, u)|}.
\end{aligned}$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore, u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T.

Therefore,

$$\begin{aligned}\rho_c(u,u^*) &= \rho_c(Su,Tu^*)\\ &\precsim \alpha\rho_c(u,u^*)+\beta\frac{\rho_c(u,Su)\rho_c(u,Tu^*)}{1+\rho_c(u,u^*)}\\ &\quad+\gamma\frac{\rho_c(u,Su)\rho_c(u,Tu^*)}{1+\rho_c(u,u^*)+\rho_c(u,Tu^*)+\rho_c(u^*,Su)}\\ &\precsim \alpha\rho_c(u,u^*).\end{aligned}$$

This implies $(1-\alpha)\rho_c(u,u^*)\precsim 0$ and hence, $(1-\alpha)|\rho_c(u,u^*)|\leq 0$.

Therefore, $\rho_c(u,u^*)=0$ and hence, $u=u^*$, which implies uniqueness. Thus u is a unique common fixed point of S and T.

Corollary 3.1. *Let (X,ρ_c) be a complete complex valued metric space and $T:X\to X$ be a self mapping satisfying the following condition:*

$$\rho_c(Tx,Ty)\precsim \alpha\rho_c(x,y)+\beta\frac{\rho_c(x,Tx)\rho_c(y,Ty)}{1+\rho_c(x,y)}+\gamma\frac{\rho_c(x,Tx)\rho_c(y,Ty)}{1+\rho_c(x,y)+\rho_c(x,Ty)+\rho_c(y,Tx)}$$

for all $x,y\in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Corollary 3.2. *Let (X,ρ_c) be a complete complex valued metric space and $T:X\to X$ be a self mapping satisfying the following condition:*

$$\rho_c(T^nx,T^ny)\precsim \alpha\rho_c(x,y)+\beta\frac{\rho_c(x,T^nx)\rho_c(y,T^ny)}{1+\rho_c(x,y)}+\gamma\frac{\rho_c(x,T^nx)\rho_c(y,T^ny)}{1+\rho_c(x,y)+\rho_c(x,T^ny)+\rho_c(y,T^nx)}$$

for all $x,y\in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof. By Corollary 3.1, we obtain $\eta\in X$ such that $T^n\eta=\eta$.

The result then follows from the fact that,

$$\begin{aligned}\rho_c(T^n\eta,\eta) &= \rho_c(TT^n\eta,T^n\eta)=\rho_c(T^nT\eta,T^n\eta)\\ &\precsim \alpha\rho_c(T\eta,\eta)+\beta\frac{\rho_c(T\eta,T^nT\eta)d(\eta,T^n\eta)}{1+\rho_c(T\eta,\eta)}\\ &\quad+\gamma\frac{\rho_c(T\eta,T^nT\eta)d(\eta,T^n\eta)}{1+\rho_c(T\eta,\eta)+\rho_c(T\eta,T^n\eta)+d(\eta,T^nT\eta)}\\ &\precsim \alpha\rho_c(T\eta,\eta)+\beta\frac{\rho_c(T\eta,T^nT\eta)d(\eta,\eta)}{1+\rho_c(T\eta,\eta)}\\ &\quad+\gamma\frac{\rho_c(T\eta,T^nT\eta)d(\eta,\eta)}{1+\rho_c(T\eta,\eta)+\rho_c(T\eta,T^n\eta)+d(\eta,T^nT\eta)}\\ &= \alpha\rho_c(T\eta,\eta)\end{aligned}$$

Therefore, $(1-\alpha)\rho_c(T\eta,\eta) \precsim 0$, this implies, $(1-\alpha)|\rho_c(T\eta,\eta)| \leq 0$, hence $\rho_c(T^n\eta,\eta) = 0$. Thus, η is a fixed point of T. On the same lines of Theorem 3.1, we can prove the uniqueness.

Theorem 3.2. *Let (X,ρ_c) be a complete complex valued metric space and $S,T : X \to X$ be self mappings satisfying the following condition:*

$$\rho_c(Sx,Ty) \precsim \alpha\rho_c(x,y) + \beta\frac{\rho_c(x,Sx)\rho_c(y,Ty)}{1+\rho_c(x,y)} + \gamma\frac{\rho_c(x,Sx)\rho_c(y,Ty)}{1+\rho_c(x,Sx)+\rho_c(y,Ty)}$$

for all $x,y \in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then S,T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point and define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$. Then,

$$\begin{aligned}
\rho_c(x_{2k+1},x_{2k+2}) &= \rho_c(Sx_{2k},Tx_{2k+1}) \\
&\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\frac{\rho_c(x_{2k},Sx_{2k})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k},x_{2k+1})} \\
&\quad +\gamma\frac{\rho_c(x_{2k},Sx_{2k})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k},Sx_{2k})+\rho_c(x_{2k+1},Tx_{2k+1})} \\
&\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\frac{\rho_c(x_{2k},x_{2k+1})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k},x_{2k+1})} \\
&\quad +\gamma\frac{\rho_c(x_{2k},x_{2k+1})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k},x_{2k+1})+\rho_c(x_{2k+1},x_{2k+2})}.
\end{aligned}$$

Following cases arises,

Case 1. If,

$$\begin{aligned}
\rho_c(x_{2k},x_{2k+1}) &\leq 1+\rho_c(x_{2k},x_{2k+1}) \text{ and} \\
\rho_c(x_{2k+1},x_{2k+2}) &\leq 1+\rho_c(x_{2k},x_{2k+1})+\rho_c(x_{2k+1},x_{2k+2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_c(x_{2k+1},x_{2k+2}) &\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\rho_c(x_{2k+1},x_{2k+2}) + \gamma\rho_c(x_{2k},x_{2k+1}) \\
\rho_c(x_{2k+1},x_{2k+2}) &\precsim \frac{\alpha+\gamma}{1-\beta}\rho_c(x_{2k},x_{2k+1})
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho_c(x_{2k+2}, x_{2k+3}) &= \rho_c(Sx_{2k+2}, Tx_{2k+1}) \\
&\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})} \\
&\quad +\gamma\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, Sx_{2k+2})+\rho_c(x_{2k+1}, Tx_{2k+1})} \\
&\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})} \\
&\quad +\gamma\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2})}
\end{aligned}$$

Since,

$$\begin{aligned}
\rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+1}) \text{ and} \\
\rho_c(x_{2k+2}, x_{2k+3}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+1}, x_{2k+2}) + \beta\rho_c(x_{2k+2}, x_{2k+3}) + \gamma\rho_c(x_{2k+1}, x_{2k+2}) \\
\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha+\gamma}{1-\beta}\rho_c(x_{2k+1}, x_{2k+2}).
\end{aligned}$$

Assume, $h = \dfrac{\alpha+\gamma}{1-\beta} < 1$, we have

$$\rho_c(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim ... \precsim h^{n+1}\rho_c(x_0, x_1).$$

For some $m > n$, we have

$$\begin{aligned}
\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m) \\
&\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1) \\
&\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).
\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0, \ as\ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove, consider

$\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$
\begin{aligned}
\rho_c(u, Su) = z \quad &\precsim \quad \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\
&\precsim \quad \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\
&\precsim \quad \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(u, Su) + \rho_c(x_{2k+1}, Tx_{2k+1})} \\
&\precsim \quad \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\end{aligned}
$$

This implies,

$$
\begin{aligned}
|\rho_c(u, Su)| \quad &\precsim \quad |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\
&\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + z + \rho_c(x_{2k+1}, x_{2k+2})|}.
\end{aligned}
$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T.

Therefore,

$$
\begin{aligned}
\rho_c(u, u^*) \quad &= \quad \rho_c(Su, Tu^*) \\
&\precsim \quad \alpha\rho_c(u, u^*) + \beta\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)} \\
&\precsim \quad \alpha\rho_c(u, u^*).
\end{aligned}
$$

This implies $(1 - \alpha)\rho_c(u, u^*) \precsim 0$ and hence, $(1 - \alpha)|\rho_c(u, u^*)| \leq 0$.

Therefore, $\rho_c(u, u^*) = 0$ and hence, $u = u^*$, which implies uniqueness. Thus, u is a unique common fixed point of S and T.

Case 2. If,

$$
\begin{aligned}
\rho_c(x_{2k}, x_{2k+1}) \quad &\leq \quad 1 + \rho_c(x_{2k}, x_{2k+1}) \text{ and} \\
\rho_c(x_{2k}, x_{2k+1}) \quad &\leq \quad 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+2}).
\end{aligned}
$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+1}, x_{2k+2}) &\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\rho_c(x_{2k+1}, x_{2k+2}) + \gamma\rho_c(x_{2k+1}, x_{2k+2})\\ \rho_c(x_{2k+1}, x_{2k+2}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k}, x_{2k+1}).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &= \rho_c(Sx_{2k+2}, Tx_{2k+1})\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, Sx_{2k+2})+\rho_c(x_{2k+1}, Tx_{2k+1})}\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2})}\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+1}) \text{ and}\\ \rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+1}, x_{2k+2}) + \beta\rho_c(x_{2k+2}, x_{2k+3}) + \gamma\rho_c(x_{2k+2}, x_{2k+3})\\ \rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Assume, $h = \dfrac{\alpha}{1-\beta-\gamma} < 1$, we have

$$\rho_c(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim ... \precsim h^{n+1}\rho_c(x_0, x_1).$$

For some $m > n$, we have

$$\begin{aligned}\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m)\\ &\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1)\\ &\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0,\ as\ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove, consider $\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$\begin{aligned}
\rho_c(u, Su) = z &\precsim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(u, Su) + \rho_c(x_{2k+1}, Tx_{2k+1})} \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\end{aligned}$$

This implies,

$$\begin{aligned}
|\rho_c(u, Su)| &\precsim |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\
&\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + z + \rho_c(x_{2k+1}, x_{2k+2})|}.
\end{aligned}$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T. Therefore,

$$\begin{aligned}
\rho_c(u, u^*) &= \rho_c(Su, Tu^*) \\
&\precsim \alpha\rho_c(u, u^*) + \beta\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)} \\
&\precsim \alpha\rho_c(u, u^*).
\end{aligned}$$

This implies $(1 - \alpha)\rho_c(u, u^*) \precsim 0$ and hence, $(1 - \alpha)|\rho_c(u, u^*)| \leq 0$. Therefore, $\rho_c(u, u^*) = 0$ and hence, $u = u^*$, which implies uniqueness. Thus, u is a unique common fixed point of S and T.

Corollary 3.3. *Let (X, ρ_c) be a complete complex valued metric space and $T : X \to X$ be a self mapping satisfying the following condition:*

$$\rho_c(Tx, Ty) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma\frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, Tx) + \rho_c(y, Ty)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Corollary 3.4. *Let (X, ρ_c) be a complete complex valued metric space and $T : X \to X$ be a self mapping satisfying the following condition:*

$$\rho_c(T^n x, T^n y) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1+\rho_c(x, y)} + \gamma\frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1+\rho_c(x, T^n x)+\rho_c(y, T^n y)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof. By Corollary 3.3, we obtain $\eta \in X$ such that $T^n\eta = \eta$.
The result then follows from the fact that,

$$\begin{aligned}
\rho_c(T^n\eta, \eta) &= \rho_c(TT^n\eta, T^n\eta) = \rho_c(T^nT\eta, T^n\eta) \\
&\precsim \alpha\rho_c(T\eta, \eta) + \beta\frac{\rho_c(T\eta, T^nT\eta)d(\eta, T^n\eta)}{1+\rho_c(T\eta, \eta)} + \gamma\frac{\rho_c(T\eta, T^nT\eta)d(\eta, T^n\eta)}{1+\rho_c(T\eta, T^nT\eta)+d(\eta, T^n\eta)} \\
&\precsim \alpha\rho_c(T\eta, \eta) + \beta\frac{\rho_c(T\eta, T^nT\eta)d(\eta, \eta)}{1+\rho_c(T\eta, \eta)} + \gamma\frac{\rho_c(T\eta, T^nT\eta)d(\eta, \eta)}{1+\rho_c(T\eta, T^nT\eta)+d(\eta, T^n\eta)} \\
&= \alpha\rho_c(T\eta, \eta).
\end{aligned}$$

Therefore, $(1-\alpha)\rho_c(T\eta, \eta) \precsim 0$, this implies, $(1-\alpha)|\rho_c(T\eta, \eta)| \leq 0$, hence $\rho_c(T^n\eta, \eta) = 0$. Thus, η is a fixed point of T. On the same lines of Theorem 3.2, we can prove the uniqueness.

4. Deduction

Theorem 4.1. [2] *[Azam et al.] Let (X, ρ_c) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\rho_c(Sx, Ty) \preceq \lambda\rho_c(x, y) + \frac{\mu\rho_c(x, Sx)\rho_c(y, Ty)}{1+\rho_c(x, y)}$$

for all $x, y \in X$, where λ, μ are non-negative reals with $\lambda+\mu<1$. Then S, T have a unique common fixed point.

Proof. The required result can be obtained by assuming $\gamma = 0$ in Theorem 3.1 and 3.2.

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International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:15-34

www.journalmim.com

CONFORMAL η-RICCI SOLITONS IN δ- LORENTZIAN TRANS SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study the δ-Lorentzian Trans Sasakian manifolds admitting the conformal η-Ricci Solitons and gradient conformal Ricci soliton. It is shown that a symmetric second order covariant tensor in a δ-Lorentzian Trans Sasakian manifold is a constant multiple of metric tensor. Also an example of conformal η-Ricci soliton in 3-dimensional δ-Lorentzian Trans Sasakian manifold is provided in the region where δ-Lorentzian Trans-Sasakian manifold expanding.

1. Introduction

In recent years the pioneering works of R. Hamilton [22] and G. Perelman [34] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future.

Received: 2018-02-20 *Revised:2018-03-07* *Accepted:2018-03-14*

2010 Mathematics Subject Classification: 53C15, 53C20, 53C25, 53C44.

Key words: Conformal η-Ricci soliton, δ-Lorentzian Trans Sasakian manifold, gradient conformal Ricci soliton, Einstein manifold.

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In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$. When X is the gradient of a potential $\psi \in C^\infty(M)$, the soliton is called a gradient Ricci soliton [13] and the previous equation (1.1) takes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.2}$$

Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$Ric = \lambda g. \tag{1.3}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X = 0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

It is well know fact that, if the potential vector filed ψ is zero or Killing then the Ricci soliton is an Einstein real hypersurfaces on non-flat complex soace forms [11]. Motivated by this in 2009, J.T. Cho and M. Kimura [12] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons.

Definition 1.2. *An η-Ricci soliton (g, V, λ, μ) on a Riemannian manifold is defined by*

$$\mathcal{L}_X g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ is a real scalar. In particular $\mu = 0$ then the data (g, ξ, λ) is a Ricci soliton.

In [19], A.E. Fischer introduced a new concept called conformal Ricci flow which is a

variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial t}{\partial t} = -2S - \left(p + \frac{2}{n}\right) g, \tag{1.6}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M.

The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.

In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g = 0, \tag{1.7}$$

where λ is a constant.

Therefore, It is an interesting and natural to see the condition in case of conformal η-Ricci soliton. From equations (1.5) and (1.7) we are introducing the notion of conformal η-Ricci soliton by the following equation

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{1.8}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ is a real scalar. In particular $\mu = 0$ then the data (g, ξ, λ) is a conformal-Ricci soliton [1]. The theory of differentiable manifolds with Lorentizain metric is a natural and interesting topic in differential geometry. In [24], T. Ikawa and M. Erdogan studied Lorentzian Sasakian manifold. Lorentzian Kenmotsu manifold introduced by Mihai et al. [29] and K. Kenmotsu [25]. Also Lorentzian para contact manifolds were introduced by K. Matsumoto [28]. Trans Lorentzian para Sasakian manifolds have been used by H. Gill and K. K. Dube [21]. In ([48]

[49]) A. Yıldız et al. studied Lorentzian α- Sasakian also Lorentzian-Sasakian manifolds and Lorentzian β-Kenmotsu manifold studied by Funda et al. in [47]. After that in 2011 S. S Pujar and V. J. Khairnar [35] have initiated the study of Lorentzian Trans-Sasakian manifolds and studied the some basic results with some of its properties. Earlier to this , S. S. Pujar [36] has initiated the study of δ-Lorentzian α Sasakian manifolds. In [16] U. C. De also studied properties of curvatures in Lorentzian Trans Sasakian manifolds.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relatively. In 1969, Takahashi [42] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost conatct metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds [46]. The concept of (ε)-Sasakian manifolds was initiated by Bejancu and Duggal [4]. U. C. De and A. Sarkar [14] studied the notion of (ε)-Kenmotsu manifolds. S.S. Shukla and D. D. Singh [38] extended the study to (ε)-Trans-Sasakian manifolds with indefnite metric. Siddiqi et al. [39] also studied some properties of Indefinite trans-Sasakian manifolds which is closely related to this topic. The semi-Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This motivated the Thripathi and others [43] to introduced (ε)-almost para contact structure where the vector filed ξ is space like or time like according as $(\varepsilon) = 1$ or $(\varepsilon) = -1$.

When M has a Lorentzian metric g, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results In 2014, S. M Bhati [2] introduced the notion of δ-Lorentzian Trans Sasakian manifolds.

In 1925, Levy [26] proved that a second order parallel symmetric non-sigular tensor in real space forms is proportional the metric tensor. Later, R. Sharma [37] initiated the study of Ricci solitons in contact Riemannian geometry . After that, many authors extensively studied Ricci soliton (see [8], [9], [10], [23], [30], [31], [40], [41]). The study of η-Ricci solitons in (ε)-almost paracontact metric manifolds have been studied by A. M. Blaga et al. [7]. Recently, A. M. Blaga and various others authors also have been studied η-Ricci solitons in manifolds

with different structures (see [5], [6], [37]). Recently K. Venu et al. [45] study the η-Ricci solitns in trans-Sasakian maanifold. In 2016, T. Dutta et al. [17] studied conformal Ricci soliton in Lorentzian α-Sasakian manifols. It is natural and interesting to study Conformal η-Ricci soliton in δ-Lorentzian Trans-Sasakian manifolds. In this paper we derive the condition for a 3 dimensional δ-Lorentzian Trans-Sasakian manifold as a confromal η-Ricci soliton and derive expression for the scalar curvature. Moreover, in the last section studied the gradient conformal Ricci soliton for a 3 dimensional δ-Lorentzian Trans-Sasakian manifolds.

2. Preliminaries

Let M be an δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{2.9}$$

$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y), \quad \eta(X) = \delta g(X, \xi), \quad g(\xi, \xi) = -\delta, \tag{2.10}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [35] on M, the vector field ξ is the time like [43], that is M contains a time like vector field. In [44], Tanno classified the connected almost contact metric manifold.

In [20], Grey and Harvella was introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [32] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely [27].

An almost contact metric structure on M is called a trans-Sasakian (see [3], [32]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.11}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature now we define the δ-Lorentzian trans-Sasakian manifolds [2] as follows.

Definition 2.1. *A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X). \tag{2.12}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [32]. δ-Lorentzian trans Sasakian manifold of type $(0, 0)$, $(0, \beta)$ $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively. Form (2.12), we have

$$\nabla_X \xi = \delta \left\{ -\alpha\phi(X) - \beta(X + \eta(X)\xi \right\}, \tag{2.13}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X,Y) + \delta\eta(X)\eta(Y)]. \tag{2.14}$$

In a δ-Lorentzian trans Sasakian manifold M, we have the following relations:

$$R(X,Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{2.15}$$

$$+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

$$S(X,\xi) = [((n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.16}$$

$$Q\xi = \delta(n-1)(\alpha^2 + \beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.17}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$. Further in an δ-Lorentzian trans Sasakian manifold , we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.18}$$

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.19}$$

Using (2.15) and (2.18), for constants α and β , we have

$$R(\xi, X)Y = (\alpha^2 + \beta^2)[\delta g(X, Y)\xi - \eta(Y)X], \tag{2.20}$$

$$R(X, Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.21}$$

$$\eta(R(X, Y)Z) = \delta(\alpha^2 + \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.22}$$

$$S(X, \xi) = [((n-1)(\alpha^2 + \beta^2) - \delta(\xi\beta)]\eta(X), \tag{2.23}$$

$$Q\xi = [(n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\xi. \tag{2.24}$$

An important consequence of (2.21) is that ξ is a geodesic vector field.

$$\nabla_\xi \xi = 0. \tag{2.25}$$

For arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0. \tag{2.26}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.21), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 + \beta^2) - \delta(\xi\beta). \tag{2.27}$$

It follows from (2.27) that ξ-sectional curvature does not depend on X.

3. CONFORMAL η-SOLITONS ON $(M, \phi, \xi, \eta, g, \delta)$

In the study of the conformal η-Ricci soliton equation we will consider certain assumptions, one essential condition being $\nabla\xi = I_\xi(M) + \eta \otimes \xi$ which naturally arises in different geometry of δ-Lorentzian trans-Sasakian manifolds.

An important geometrical object in studying Ricci solitons is a symmetric $(0, 2)$- tensor field which is parallel with respect to the Levi-Civita connection
Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [18].

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{3.28}$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{3.29}$$

Replacing $Z = W = \xi$ in (3.29) and using (2.15) and also use the symmetry of h, we have

$$2(\alpha^2 + \beta^2)[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] + 2\delta[(Y\alpha)h(\phi X, \xi) - (X\alpha)h(\phi Y, \xi)] \tag{3.30}$$

$$+2\delta[(Y\beta)h(\phi^2X,\xi)-(X\beta)h(\phi^2Y,\xi)]+4\alpha\beta[\eta(Y)h(\phi X,\xi)-\eta(X)h(\phi Y,\xi)]$$

Putting $X=\xi$ in (3.30) and by virtue of (2.9), we obtain

$$-2[(\delta\xi\alpha-2\alpha\beta]h(\phi Y,\xi)+2[(\alpha^2+\beta^2)-\delta(\xi\beta)][\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{3.31}$$

By using (2.19) in (3.31), we have

$$[(\alpha^2+\beta^2)-\delta(\xi\beta)][\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{3.32}$$

Suppose $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, it results

$$h(Y,\xi)=\eta(Y)h(\xi,\xi). \tag{3.33}$$

Now, we can call a regular δ-Lorentzian trans Sasakian manifold with $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of δ-Lorentzian trans Sasakian manifolds. Differentiating (3.33) covariantly with respect to X, we have

$$(\nabla_X h)(Y,\xi)+h(\nabla_X Y,\xi)+h(Y,\nabla_X\xi)=[\delta g(\nabla_X Y,\xi)+\delta g(Y,\nabla_X\xi)]h(\xi,\xi) \tag{3.34}$$

$$+\eta(Y)[(\nabla_X h)(Y,\xi)+2h((\nabla_X\xi,\xi)].$$

By using the parallel condition $\nabla h=0$, $\eta(\nabla_X\xi)=0$ and by the virtue of (3.33) in (3.34), we get

$$h(Y,\nabla_X\xi)=\delta g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now using (2.13) in the above equation, we get

$$-\alpha h(Y,\phi X)+\beta\delta h(Y,X)=-\alpha g(Y,\phi X)h(\xi,\xi)+\beta\delta g(Y,X)h(\xi,\xi). \tag{3.35}$$

Replacing $X=\phi X$ in (3.35) and after simplification, we get

$$h(X,Y)=\delta g(X,Y)h(\xi,\xi), \tag{3.36}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.33). Now by considering the above equations, we can gives the conclusion:

Theorem 3.1. *Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-Lorentzian trans Sasakian manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h\in\Gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

Definition 3.1. *Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-almost contact metric manifold. consider the equation*

$$\mathcal{L}_\xi g+2S+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g+2\mu\eta\otimes\eta=0, \tag{3.37}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. For $\mu\neq 0$, the data (g,ξ,λ,μ) will be called conformal Ricci-soliton.

Remark 3.1. *If the scalar curvature $-\frac{1}{2}(p+\frac{2}{n})$ of the manifold is constant, then the conformal η-Ricci soliton $(g,\xi,\left\{\lambda-\frac{1}{2}(p+\frac{2}{n})\right\},\mu)$ reduces to an η-Ricci soliton and, moreover, if $\mu=0$, to a Ricci soliton $(g,\xi,\left\{\lambda-\frac{1}{2}(p+\frac{2}{n})\right\})$. Therefore, the two concepts of Conformal η-Ricci soliton and η-Ricci soliton are distinct on manifolds of non constant scalar curvature.*

Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain [13]:

$$2S(X,Y)=-g(\nabla_X\xi,Y)-g(X,\nabla_X\xi)-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)-2\mu\eta(X)\eta(Y), \tag{3.38}$$

for any $X,Y\in\chi(M)$.

The data (g,ξ,λ,μ) which satisfy the equation (3.37) is said to be conformal η- Ricci soliton on M [12] and its called shrinking, steady or expanding according as $\lambda<0$, $\lambda=0$ or $\lambda>0$ respectively [12]. Now, from (2.13) , the equation (3.37) becomes:

$$S(X,Y)=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)+\delta\beta\right]g(X,Y)+(\beta\delta-\mu)\eta(X)\eta(Y). \tag{3.39}$$

The above equations yields

$$S(X,\xi)=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)+\mu\right]\eta(X) \tag{3.40}$$

$$QX=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)+\delta\beta\right]X+(\beta\delta-\mu)\xi \tag{3.41}$$

$$Q\xi=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)+\mu\right]\xi \tag{3.42}$$

$$r=-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)+\delta\beta\right]n-(n-1)\beta\delta-\mu, \tag{3.43}$$

where r is the scalar curvature. Of the two natural situations regrading the vector field V: $V\in Span\{\xi\}$ and $V\perp\xi$, we investigate only the case $V=\xi$.

Our interest is in the expression for $\mathcal{L}_\xi g+2S+2\mu\eta\otimes\eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y)=2\beta\delta[g(X,Y)+\eta(X)\eta(Y)]. \tag{3.44}$$

In 3-dimensional δ-Lorentzian trans Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \tag{3.45}$$
$$- \frac{r}{2}[g(Y,Z)X - g(X,Z)Y],$$

Putting $Z = \xi$ in (3.45) and using (2.15) and (2.16) for 3-dimensional δ-Lorentzian trans-Sasakian manifold, we get

$$(\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{3.46}$$
$$+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$
$$= [(\alpha^2 + \beta^2) - (\xi\beta)][\eta(Y)X - \eta(X)Y]$$
$$+ \delta\eta(Y)QX - \delta\eta(X)QY - \delta[((\phi Y)\alpha)X + (Y\beta)X]$$
$$+\delta[((\phi X)\alpha)Y + (X\beta)Y].$$

Again, putting $Y = \xi$ in the (3.46) and using (2.9) and (2.19), we obtain

$$QX = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2 + \beta^2)\right] X + \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2 + \beta^2)\right] \eta(X)\xi. \tag{3.47}$$

From (3.47), we have

$$S(X,Y) = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2 + \beta^2)\right] g(X,Y) \tag{3.48}$$
$$+ \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2 + \beta^2)\right] \delta\eta(X)\eta(Y).$$

Equation (3.48) shows that a 3-dimensional (ϵ, δ)-trans-Sasakian manifold is η-*Einstein*. Next, we consider the equation

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y). \tag{3.49}$$

By Using (3.44) and (3.48) in (3.49), we have

$$h(X,Y) = \left[r - 4(\alpha^2 + \beta^2) + 2\beta\delta\right] g(X,Y) \tag{3.50}$$
$$+ \left[8(\alpha^2 + \beta^2) - 2\beta\delta - r\right] \delta\eta(X)\eta(Y) + 2\mu\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in (2.11), we get

$$h(\xi,\xi) = 2[2\delta(\alpha^2 + \beta^2) - 2\mu] \tag{3.51}$$

Now, (3.36) becomes

$$h(X,Y) = 2[2\delta(\alpha^2 + \beta^2) - 2\mu]\delta g(X,Y). \tag{3.52}$$

From (3.49) and (3.52), it follows that g is conformal η-Ricci soliton.

Therefore, we can state as:

Theorem 3.2. *Let $(M, \phi, \xi, \eta, g, \delta)$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold, then $(g, \xi, \left\{\lambda - \frac{1}{2}(p + \frac{2}{n})\right\}, \mu)$ yields a conformal η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional δ-Lorentzian trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

or

$$\begin{aligned} bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) \\ +2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned}$$

Using (2.13), we obtain

$$\begin{aligned} bg(-\delta\alpha\phi X - \beta\delta(X + \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(-\delta\alpha\phi Y - \beta\delta(Y + \eta(Y)\xi, X) \\ +(Yb)\eta(X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned}$$

which yields

$$\begin{aligned} -2b\beta\delta g(X,Y) - 2b\beta\delta\eta(X)\eta(Y) + (Xb)\eta(Y) \\ +(Yb)\eta(X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.53}$$

Replacing Y by ξ in (3.53), we obtain

$$(Xb) + (\xi b)\eta(X) + 2\left[2(\alpha^2 + \beta^2) - (\xi\beta) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - 2b\beta\delta\right]\eta(X). \tag{3.54}$$

Again putting $X = \xi$ in (3.54), we obtain

$$\xi b = -2(\alpha^2 + \beta^2) + (\xi\beta) - \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] - \mu + 2b\beta\delta.$$

Plugging this in (3.54), we get

$$(Xb) + 2[2(\alpha^2 + \beta^2) - (\xi\beta) - \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - 2b\beta\delta]\eta(X) = 0,$$

or

$$db = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2((\alpha^2 + \beta^2) - 2b\beta\delta)\eta. \tag{3.55}$$

Applying d on (3.55), we get $\left\{-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta\right\} d\eta$. Since $d\eta \neq 0$ we have

$$-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta = 0. \tag{3.56}$$

Equation(3.56) in (3.55) yields b as a constant. Therefore from (3.53), it follows that

$$S(X,Y)=\left(-\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+2b\beta\delta\right)g(X,Y)+(2b\beta\delta-\mu)\eta(X)\eta(Y), \tag{3.57}$$

which implies that M is of constant scalar curvature for constant $2b\beta\delta$. This leads to the following:

Theorem 3.3. *If in a 3-dimensional δ-Lorentzian trans-Sasakian manifold the metric g is a conformal η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $b\beta\delta$ is a constant.*

Tanking $X=Y=\xi$ in (3.36) and (3.48) and comparing, we get

$$\lambda=\frac{1}{2}\left(p+\frac{2}{n}\right)-2(\alpha^2+\beta^2)-\delta(\xi\beta)+\mu-2b\beta\delta=-2K_\xi+\frac{1}{2}\left(p+\frac{2}{n}\right)-\mu. \tag{3.58}$$

From (3.43) and (3.57) also put $n=3$, we obtain

$$r=\left(\frac{p}{2}+\frac{1}{3}\right)+6(\alpha^2+\beta^2)-3\delta(\xi\beta)-2\beta\delta+2\mu. \tag{3.59}$$

Now for conformal Ricci soliton $r=-1$, so putting this value in the above equation we get

$$\mu=-\left(p+\frac{2}{3}\right)-(\alpha^2+\beta^2)+\frac{3}{2}\delta(\xi\beta)+\beta\delta.$$

Since λ is a constant, it follows from (3.57) that K_ξ is a constant.

Theorem 3.4. *Let (g,ξ,μ) be a conformal η-Ricci soliton in $(M,\phi,\xi,\eta,g,\delta)$ a 3-dimensional δ-Lorentzian trans-Sasakian manifold. Then the scalar $\lambda-\left(\frac{p}{2}+\frac{1}{3}\right)+\mu=-2K_\xi$, $r=6K_\xi+2\mu-3(\xi\beta)-2b\beta\delta+\left(\frac{p}{2}+\frac{1}{3}\right)$.*

Remark 3.2. *For $\mu=0$, (3.57) reduces to $\lambda=-2K_\xi+\left(\frac{p}{2}+\frac{1}{3}\right)$, so confromal Ricci soliton in 3-dimensional δ-Lorentzian trans-Sasakian manifold is shrinking.*

Example 3.1. *Consider the 3-dimensional manifold $M=\left\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\right\}$, where (x,y,z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the vector fields are*

$$e_1=\frac{e^x}{z^2}\frac{\partial}{\partial x},\quad e_2=\frac{e^y}{z^2}\frac{\partial}{\partial y},\quad e_3=\frac{-(\delta)}{2}\frac{\partial}{\partial z},$$

where e_1,e_2,e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by

$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=-\delta$, $g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0$,

where δ is such that $\delta^2=1$ so that $\delta=\pm 1$.

Let η be the 1-form defined by $\eta(X) = \delta g(X, \xi)$ for any vector field X on M, and ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (δ)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is know as Koszul's formula.

$\nabla_{e_1} e_3 = -\frac{\delta}{z} e_1, \quad \nabla_{e_2} e_3 = -\frac{\delta}{z} e_2, \quad \nabla_{e_1} e_2 = 0,$
using the above relation, for any vector X on M, we have $\nabla_X \xi = \delta[-\alpha\phi X - \beta(X + \eta(X)\xi)]$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the δ-Lorentzian trans-Sasakian structure in $\mathbb{R}^3$.

Here ∇ be the Levi-Civita connection with respect to the metric g , then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -\frac{(\delta)}{z} e_1, \quad [e_2, e_3] = -\frac{(\delta)}{z} e_2.$$

Since $g(e_1, e_2) = 0$. Thus we have

$$\nabla_{e_1} e_3 = -\frac{(\delta)}{z} e_1 + e_2, \quad \nabla_{e_1} e_2 = 0$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\frac{(\delta)}{z} e_2, \quad \nabla_{e_2} e_3 = -\frac{(\delta)}{z} e_2 - e_1$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -\frac{(\delta)}{z} e_1 + e_2.$$

The manifold M satisfies (2.5) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is an δ-Lorentzian trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1, e_3)e_3 = \frac{(\delta)}{z^2} e_1, \quad R(e_3, e_1)e_3 = -\frac{(\delta)}{z^2} e_1,$$

$$R(e_2, e_3)e_3 = \frac{(\delta)}{z^2} e_1, \quad R(e_3, e_2)e_3 = -\frac{(\delta)}{z^2} e_1.$$

From the above expression of the curvature tensor we can also obtain the Ricci tensor

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \frac{(\delta^2)}{z^2}$$

since $g(e_1, e_3) = g(e_1, e_2) = 0$.

Therefore, we have

$$S(e_i, e_i) = \frac{(\delta)}{z^2} g(e_i, e_i),$$

and the scalar curvature $scal = 3\frac{(\delta^2)}{z^2}$. for $i = 1, 2, 3$, and $\alpha = \frac{1}{z}$, $\beta = -\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (3.11), computed (e_i, e_i) as follows

$$2[g(e_i, e_i) - \eta(e_i)\eta(e_i)] + 2S(e_i, e_i) + \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i) = 0$$

for all $i \in \{1, 2, 3\}$, and we have

$$2(1 - \delta_{i3}) + 2\frac{\delta}{z^2} + (2\lambda - 3\frac{\delta}{z^2}) + 2\mu\delta_{i3} = 0$$

for all $i \in \{1, 2, 3\}$

Therefore $\lambda = 2\left(\frac{p}{4} - \frac{1}{3} - \frac{(\delta)}{z^2}\right)$ and $\mu = -\frac{(\delta)}{z^2} + 1$, the data (g, ξ, λ, μ) is an conformal η-Ricci soliton on $(M, \phi, \xi, \eta, g, \delta)$.

Here in this example if $\mu = 0$, then (g, ξ, λ, μ) reduce to conformal Ricci soliton for $\lambda = 2\left(\frac{p}{4} - \frac{1}{3} - \frac{(\delta)}{z^2}\right)$ which is positive. Therefore conformal Ricci soliton is expanding for $\lambda > 0$.

4. Gradient Conformal Ricci Solitons in 3-dimensional δ-Lorentzian trans-Sasakian

Definition 4.1. *A Riemannian manifold (M, g) is said to be conformal gradient Ricci soliton if there exist a confromal change of the metric $\bar{g} = e^u g$, $u \in C^\infty(M)$, a function $\psi \in C^\infty(M)$ and a constant $\lambda \in \mathbb{R}$ such that*

$$Ric + Hess(\psi) = \lambda\bar{g} \tag{4.60}$$

If the vector field V is the gradient of a potential function -ψ then $\bar{g}$ is called a conformal gradient Ricci soliton and (1.2) assume the form

$$\nabla\nabla\psi = S + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] e^u g. \tag{4.61}$$

This reduces to

$$\nabla_Y D\psi = QY + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] Y, \tag{4.62}$$

where D denoted the gradient operator of g. From (4.61) it follows

$$R(X, Y)D\psi = (\bar{\nabla}_X Q)Y - (\bar{\nabla}_Y Q)X. \tag{4.63}$$

Differentiating (3.47) we get

$$(\nabla_W Q)X = \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3(\alpha^2 + \beta^2))(\alpha(g(\phi W, X) \tag{4.64}$$

$$+\beta\delta g(W, X) - \delta\beta\eta(X)\eta(W)) + \eta(X)\nabla_W\xi.$$

In (4.63) replacing $W = \xi$, we obtain

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \tag{4.65}$$

Then we have

$$g(\nabla_\xi Q)X - (\bar{\nabla}_X Q)(\xi, \xi) \tag{4.66}$$

$$= g(\frac{dr(\xi)}{2}(X - \eta(X)\xi, \xi)) = \frac{dr(\xi)}{2}(g(X, \xi) - \eta(X))) = 0.$$

Using (4.65) and (4.64), we obtain

$$g(R(\xi, X)D\psi, \xi) = 0. \tag{4.67}$$

From (2.20)

$$g(\bar{R}(\xi, Y)D\psi, \xi) = (\alpha^2 + \beta^2)(g(Y, D\psi) - \eta(Y)\eta(D\psi)).$$

Using (4.66), we get

$$(\alpha^2 + \beta^2)(g(Y, D\psi) - \eta(Y)\eta(D\psi)) = 0$$

$$(\alpha^2 + \beta^2)(g(Y, D\psi) - \eta(Y)g(D\psi, \xi)) = 0,$$

or

$$(g(Y, D\psi) - g(Y, \xi)g(D\psi, \xi)) = 0,$$

which implies

$$(g(Y, D\psi) - g(Y, \xi)g(D\psi, \xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \quad \text{since} \quad \alpha^2 + \beta^2 \neq -\delta(\xi\beta). \tag{4.68}$$

Using (4.67) and (4.61)

$$S(X, Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] e^u g(X, Y) = g(\nabla_Y D\psi, X) = g(\nabla_Y(\xi\psi)\xi, X)$$

$$= (\xi\psi)g(\bar{\nabla}_Y\xi, X) + Y(\xi\psi)\eta(X)$$

$$= (\xi\psi)g(-\delta\alpha\phi Y - \delta\beta Y - \delta\beta\eta(Y)\xi, X) + Y(\xi\psi)\eta(X)$$

$$S(X, Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right]\bar{g}(X, Y) = -\delta\alpha(\xi\psi)g(\phi Y, X) - \delta\beta(\xi\psi)\bar{g}(Y, X) \tag{4.69}$$

$$-\delta\beta(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X).$$

Putting $X=\xi$ in (4.68) and using (2.23) we get

$$\bar{S}(Y,\xi)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]e^{u}\eta(Y)=Y(\xi\psi)=[\lambda+2\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))]e^{u}\eta(Y). \quad (4.70)$$

Interchanging X and Y in (4.68), we get

$$S(X,Y)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-\delta\alpha(\xi\psi)g(Y,\phi X) \quad (4.71)$$

$$-\delta\beta(\xi\psi)\bar{g}(X,Y)-\delta\beta(\xi\psi)\eta(Y)\eta(X)+X(\xi\psi)\eta(Y).$$

Adding (4.68) and (4.70) we get

$$2S(X,Y)+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-2\delta\beta(\xi\psi)\bar{g}(X,Y)+Y(\xi\psi)\eta(X) \quad (4.72)$$

$$-2\delta\beta(\xi\psi)\eta(X)\eta(Y)+X(\xi\psi)\eta(Y).$$

Using (4.69) in (4.71) we have

$$S(X,Y)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-\delta\beta(\xi\psi)[g(X,Y)-\eta(X)\eta(Y)] \quad (4.73)$$

$$+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))]\eta(X)\eta(Y).$$

Then using (4.61) we have

$$\nabla_{Y}D\psi=-\delta\beta(\xi\psi)(Y-\eta(Y)\xi) \quad (4.74)$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))\right]\eta(Y)\xi.$$

Using (4.73) we calculate

$$R(X,Y)D\psi=\nabla_{X}\nabla_{Y}D\psi-\nabla_{Y}\nabla_{X}D\psi-\nabla_{[X,Y]}D\psi$$

$$=-\delta\beta X(\xi\psi)Y+\delta\beta Y(\xi\psi)X \quad (4.75)$$

$$-\delta\beta Y(\xi\psi)\eta(X)\xi+\delta\beta X(\xi\psi)\eta(Y)\xi$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))\right]((\nabla_{X}\eta)(Y)\xi-(\nabla_{Y}\eta)(X)\xi)$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))\right]((\nabla_{X}\xi)\eta(Y)\xi-(\nabla_{Y}\xi)\eta(X)).$$

Taking inner product with ξ in (4.74), we get

$$0=g((X,Y)D\psi,\xi)=2\delta\alpha+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))\right]g(\phi Y,X). \quad (4.76)$$

Thus we have $2\delta\alpha+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^{2}+\beta^{2}-\delta(\xi\beta))\right]=0.$

Now we consider the following cases:

Case (i) $\delta\alpha = 0$, or

Case (ii) $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$,

Case (iii) $\alpha = 0$ and $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$.

Case (i) If $\alpha = 0$, the manifold reduces to a δ-Lorentzian β-Kenmotsu manifold.

Case (ii) Let $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$. If we use this in (4.69) we get $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Substitute this value in (4.71) we obtain

$$S(X,Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) = -\delta\beta(\xi\psi)g(X,Y) - 2\delta\beta\eta(X)\eta(Y). \tag{4.77}$$

Now, contracting (4.76), we get

$$r + \frac{3}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] = -3\delta\beta(\xi\psi) - 2\delta\beta. \tag{4.78}$$

Putting $n = 3$ and for conformal Ricci soliton $r = -1$ in (4.78) which implies

$$(\xi\psi) = -\frac{1}{-\delta\beta}\left(\lambda + \frac{p}{2}\right) - \frac{2}{3}. \tag{4.79}$$

If $r = -1$, then $(\xi\psi) = constant = k(say)$. Therefore from (4.67) we have $D\psi = (\xi\psi)\xi = k\xi$. This we can write this equation as

$$g(D\psi, X) = k\eta(X), \tag{4.80}$$

which means that $d\psi(X) = k\eta(X)$. Applying d this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$ Therefore equation (4.60) reduces to

$$S(X,Y) = 2(\alpha^2 + \beta^2 - \delta(\xi\beta))g(X,Y),$$

that is M is an *Einstein* manifold.

Case (iii) Using $\alpha = 0$ and $\left[\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))\right] = 0$. in (4.69) we obtain $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.

Thus we have the following :

Theorem 4.1. *If a 3-dimensional δ-Lorentzian trans Sasakian manifold with constant scalar curvature admits gradient Einstein soliton, then the manifold is either a δ-Lorentzian β-Kenmotsu manifold or an Einstein manifold provided $\alpha, \beta = constant$.*

In [15] it was proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 (see [15]) we state the following:

Corollary 4.1. *If a compact 3-dimensional δ-Lorentzian trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either δ-Lorentzian α-Sasakian or δ-Lorentzian β-Kenmotsu.*

Also in [15], authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 in [15], we obtain the following:

Corollary 4.2. *If a locally ϕ-symmetric 3-dimensional connected δ-Lorentzian trans-Sasakian manifold its admits gradient conformal soliton, then manifold is either δ-Lorentzian β-Kenmotsu or Einstein manifold provided $\alpha, \beta = constant$.*

Acknowledgement

The author would like to thank the referee for very careful reading and many helpful comments which have improved the paper.

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International Journal of Maps in Mathematics
Volume:1, Issue:1, (2018), Pages:35-49
www.journalmim.com

FINITE TIME BLOW-UP FOR FRACTIONAL TEMPORAL SCHRÖDINGER EQUATIONS AND SYSTEMS ON THE HEISENBERG GROUP

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Abstract. The aim of this research paper is to establish sufficient conditions for the nonexistence of global weak solution to the nonlinear Schrödinger equation on the Heisenberg group. The results are shown by the use of test function theory and extended to systems of the same type.

1. Introduction

The main purpose of this paper is to present results concerning the local nonexistence of solutions for the following nonlinear time fractional Schrödinger equation posed in Heisenberg group

$$i^{\alpha}\ {}_{0}^{C}D_{t}^{\alpha}u+\Delta_{\mathbb{H}}u=\lambda|u|^{p}+\mu a\left(\eta\right).\nabla_{\mathbb{H}}|u|^{q}, \tag{1.1}$$

equipped with the initial data

$$u\left(\eta,0\right)=g\left(\eta\right),$$

where $u\left(\eta,t\right)$ is a complex-valued function, $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group, $0<\alpha<1$, i^{α} is the principal value of i^{α}, ${}_{0}^{C}D_{t}^{\alpha}$ is the Caputo fractional derivative of order α, $\lambda=\lambda_1+i\lambda_2$, $(\lambda_1,\lambda_2)\in\mathbb{R}^2-\{(0;0)\}$,

Received: 2018-03-03 *Accepted:2018-03-14*

2010 Mathematics Subject Classification: 35A01, 35R03, 35D02.

Key words: Blow-up, Caputo fractional derivative, Schrödinger equations

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$\mu = \mu_1 + i\mu_2$, $(\mu_1, \mu_2) \in \mathbb{R}^2$ and $p > q > 1$. The symbol $\nabla_{\mathbb{H}}$ denotes the gradient over $\mathbb{H}$ and $a(\eta) = (A_1(\eta); A_2(\eta); ...; A_N(\eta)) \in \mathbb{R}^N$ is a given vector function, assumed to satisfy

$$|a(T^{\frac{Q\alpha}{2}}\tilde{\eta})| \simeq T^{\nu}, \qquad |Div_{\mathbb{H}}(a(T^{\frac{Q\alpha}{2}}\tilde{\eta}))| \simeq T^{\tau}. \tag{1.2}$$

Therefore $a(\eta).\nabla_{\mathbb{H}}|u|^q$ is the scalar product of $a(\eta)$ and $\nabla_{\mathbb{H}}|u|^q$ and $g(\eta) = g_1(\eta) + ig_2(\eta)$, $(g_1(\eta); g_2(\eta)) \in \mathbb{R}^2$, $g \in L^1(\mathbb{H})$. Then we extend our analysis to the 2×2 system:

$$\begin{cases} i^{\alpha}\, {}^{c}_{o} D^{\alpha}_{t} u + \Delta_{\mathbb{H}} u = \lambda |v|^p + \mu a(\eta).\nabla_{\mathbb{H}}|v|^q \\ i^{\beta}\, {}^{C}_{0} D^{\beta}_{t} v + \Delta_{\mathbb{H}} v = \lambda |u|^k + \mu b(\eta).\nabla_{\mathbb{H}}|u|^{\sigma} \\ u(\eta, 0) = g(\eta); \qquad v(\eta, 0) = h(\eta), \end{cases} \tag{1.3}$$

where $0 < \beta \le \alpha < 1$; $k > \sigma > 1$. The vector functions $a(\eta) = (A_1(\eta); A_2(\eta); ...; A_N(\eta))$ and $b(\eta) = (B_1(\eta); B_2(\eta); ...; B_N(\eta))$ are assumed to satisfy

$$\begin{aligned} &|a(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta})| \simeq T^{\nu_1}, \qquad |Div_{\mathbb{H}}(a(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta}))| \simeq T^{\tau_1}, \\ &|b(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta})| \simeq T^{\nu_2}, \qquad |Div_{\mathbb{H}}(b(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta}))| \simeq T^{\tau_2}. \end{aligned} \tag{1.4}$$

Our method of proof relies on a method due to Baras and Pierre [4]. It had been remained dormant until Zhang ([16], [17], [18]) revived it. Later, this method has been successfully applied in a great number of situations by Mitidieri and Pohozaev [12] and Hakem et al [8]. This work is organized as follows. In Section 2, we present some fundamental and basic results. In section 3, we prove our main results.

2. Preliminaries

For the reader convenience, some background facts used in the sequel are recalled.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y))$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$.

This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2 \right).$$

Observe that the vector field $T = \frac{\partial}{\partial \tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in 1, 2, 3, ..., N.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$. The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = (\Delta_{\mathbb{H}} u)(\eta \circ \eta'), \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda, \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [**?**]

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

The gradient $\nabla_{\mathbb{H}}$ over $\mathbb{H}$ is defined by

$$\nabla_{\mathbb{H}} = (X_1; X_2; ...X_N; Y_1; Y_2; ...; Y_N).$$

Let

$$M = \begin{pmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \end{pmatrix}$$

where I_N is the identity matrix of size N. Then

$$\nabla_{\mathbb{H}} = M \nabla_{\mathbb{R}^{2N+1}}.$$

A simple computation gives the expression

$$|\nabla_{\mathbb{H}}|^2 = 4(|x|^2 + |y|^2) \left(\frac{\partial u}{\partial \tau} \right)^2 + \sum_{i=1}^{N} \left(\left(\frac{\partial u}{\partial x_i} \right)^2 + \left(\frac{\partial u}{\partial y_i} \right)^2 + 4 \frac{\partial u}{\partial \tau} \left(y_i \frac{\partial u}{\partial x_i} - x_i \frac{\partial u}{\partial y_i} \right) \right).$$

The divergence operator in $\mathbb{H}$ is defined by

$$Div_{\mathbb{H}}(u) = Div_{\mathbb{R}^{2N+1}}(Mu).$$

To derive the nonexistence of results for the problem (1.1), we shall state some results about fractional derivative and fractional integral which will be used in the proof of our main results (see for instance [10], [14]).

Let $f \in L^1(0;T),\ T>0$, be a given function. The Riemann-Liouville left-sided fractional integral ${}_0I_t^{\alpha}f$ of order $\alpha>0$ is defined by

$$\left({}_0I_t^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds; \quad for \quad a.e. \quad t\in[0;T],$$

where Γ is the Gamma function.

The Riemann-Liouville right-sided fractional integral ${}_tI_T^{\alpha}f$ of order $\alpha>0$ is defined by

$$\left({}_tI_T^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}f(s)ds; \quad for \quad a.e. \quad t\in[0;T].$$

Let $0<\alpha<1$ and $f\in AC^1[0;T],\ T>0$. The Caputo left-sided and right-sided fractional derivatives of order α of f are defined, respectively, by

$$\left({}_0^CD_t^{\alpha}f\right)(t)={}_0I_t^{1-\alpha}f'(t) \quad for \quad a.e. \quad t\in[0;T],$$

and

$$\left({}_t^CD_T^{\alpha}f\right)(t)=-{}_tI_T^{1-\alpha}f'(t) \quad for \quad a.e. \quad t\in[0;T].$$

The following fractional integration by parts will be used later to define the weak solutions to (1.1) and (1.3).

Lemma 2.1. *Let $0<\alpha<1$. If $f\in C[0;T]$, ${}_0^CD_t^{\alpha}f\in L^1(0;T)$, $g\in C^1[0;T]$ and $g(T)=0$, then*

$$\int_0^T \left({}_0^CD_t^{\alpha}f\right)(t)g(t)dt=\int_0^T (f(t)-f(0))(t)\left({}_t^CD_T^{\alpha}g\right)(t)dt.$$

The following results will be used several times.

Lemma 2.2. *Let $T>0$, $r\geq 1$ and $f:[0;T]\longrightarrow\mathbb{R}$ be the function given by*

$$f(t)=\left(1-\frac{t}{T}\right)^r, \qquad 0\leq t\leq T.$$

Then, for any $0<\alpha<1$, we have

$$\left({}_t^CD_T^{\alpha}f\right)(t)=\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}T^{-r}(T-t)^{r-\alpha}, \qquad 0\leq t\leq T.$$

Given a complex number $z\in\mathbb{C}$. We denote by Rez its real part and by Imz its imaginary part.

Lemma 2.3. [1] *Let $f \in \mathcal{L}^1\left(\mathbb{R}^{2N+1}\right)$ and $\int_{\mathbb{R}^{2N+1}} f d\eta > 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that*

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0.$$

Let us set $\mathcal{H}_T = \mathbb{H} \times (0,T)$ and $\mathcal{H} = \mathbb{H} \times (0,\infty)$ for $T > 0$. We put

$$G_1(\alpha; g(\eta)) = cos\left(\frac{\alpha\pi}{2}\right) g_1(\eta) - sin\left(\frac{\alpha\pi}{2}\right) g_2(\eta);$$
$$G_2(\alpha; g(\eta)) = cos\left(\frac{\alpha\pi}{2}\right) g_2(\eta) + sin\left(\frac{\alpha\pi}{2}\right) g_1(\eta).$$

3. Main results

3.1. Case of a single equation.

Definition 3.1. *A locally integrable function $u \in L^{max\{p;q\}}_{loc}(\mathcal{H}_\mathcal{T})$ is called a local weak solution to (1.1) in $\mathcal{H}_\mathcal{T}$ subject to the initial data $g \in L^1(\mathbb{H})$ if the equality*

$$\begin{aligned} &\lambda \int_{\mathcal{H}_\mathcal{T}} |u|^p \varphi d\eta dt + i^\alpha \int_{\mathcal{H}_\mathcal{T}} g(\eta) \, {}^C_t D^\alpha_T \varphi d\eta dt = \int_{\mathcal{H}_\mathcal{T}} u \left(i^\alpha \, {}^C_t D^\alpha_T \varphi + \Delta_\mathbb{H} \varphi \right) d\eta dt \\ &+ \mu \int_{\mathcal{H}_\mathcal{T}} |u|^q \varphi div_\mathbb{H}(a(\eta)) d\eta dt + \mu \int_{\mathcal{H}_\mathcal{T}} a(\eta).\nabla_\mathbb{H} \varphi |u|^q d\eta dt \end{aligned} \tag{3.5}$$

is satisfied for every test function $\varphi \in C^{1,2}_{t,\eta}(\mathcal{H}_\mathcal{T})$ with $\varphi(.,T) = 0$.
Moreover, if $T > 0$ can be arbitrarily chosen, then u is said to be a global weak solution to (1.1).

Now, we are in position to announce our first result:

Theorem 3.1. *Let $p > q > 1$ and $g \in L^1(\mathbb{H})$. Suppose that one of the following cases holds:*
(I)

$$\lambda_1 \int_\mathbb{H} G_1(\alpha; g(\eta)) d\eta > 0, \tag{3.6}$$

and

$$\mu_1 = 0, \qquad 1 < p < 1 + \frac{1}{N+1},$$

or

$$\mu_1 \neq 0, \qquad N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha - 2\nu}{2\alpha(p-q)}\right\}.$$

(II)

$$\lambda_2 \int_\mathbb{H} G_2(\alpha; g(\eta)) d\eta > 0, \tag{3.7}$$

and

$$\mu_2 = 0, \qquad 1 < p < 1 + \frac{1}{N+1},$$

or

$$\mu_2 \neq 0, \qquad N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha - 2\nu}{2\alpha(p-q)}\right\}.$$

Then the problem (1.1) admits no global weak solution.

For simplicity, we use C to denote a positive constant which may vary from line to line.

Proof. The proof is by contradiction. For that, let u be a solution and φ be a smooth nonnegative test function such that:

$$\varphi(\eta; t) = \varphi_1(t)\varphi_2(\eta), \tag{3.8}$$

where for $T > 0$, we take

$$\varphi_1(t) = \left(1 - \frac{t}{T}\right)^m, \qquad \varphi_2(\eta) = \Phi^\omega\left(\frac{\tau^2 + |x|^4 + |y|^4}{T^{2\alpha}}\right),$$

where $\omega >> 1$, $m > max\left\{1; \dfrac{\alpha p}{p-1}\right\}$ and $\phi \in C_0^\infty(\mathbb{R}^{\mathrm{N}})$ be a cut-off nonincreasing function such that

$$\Phi\left(r\right) = \begin{cases} 1 \ , \ 0 \leq r \leq 1 \\ \searrow \ , \ 1 \leq r \leq 2 \\ 0 \ , \ r \geq 2. \end{cases} \tag{3.9}$$

Let us set

$$\rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^{2\alpha}}.$$

Then we have

$$\begin{aligned} \Delta_{\mathbb{H}}\Phi^\omega &= \tfrac{4\omega(N+4)}{T^{2\alpha}}\left(|x|^2 + |y|^2\right)\Phi'\Phi^{\omega-1} \\ &+ \tfrac{16\omega}{T^{4\alpha}}\left(\left(|x|^6 + |y|^6\right) + 2\tau\left(|x|^2 - |y|^2\right)x.y + \tau^2\left(|x|^2 + |y|^2\right)\right)\Phi''\Phi^{\omega-1} \\ &+ \tfrac{16\omega(\omega-1)}{T^{4\alpha}}\left(\left(|x|^6 + |y|^6\right) + 2\tau\left(|x|^2 - |y|^2\right)x.y + \tau^2\left(|x|^2 + |y|^2\right)\right)\Phi'^2\Phi^{\omega-2}. \end{aligned} \tag{3.10}$$

Using the formula 3.5 we get

$$\begin{aligned} &Re\left\{\lambda\int_{\mathcal{H}_T}|u|^p\varphi d\eta dt + i^\alpha\int_{\mathcal{H}_T}g(\eta)\ {}_t^C D_T^\alpha\varphi d\eta dt\right\} = Re\left\{\mu\int_{\mathcal{H}_T}a(\eta).\nabla_{\mathbb{H}}\varphi|u|^q d\eta dt\right\} \\ &+ Re\left\{\int_{\mathcal{H}_T}u\left(i^\alpha\ {}_t^C D_T^\alpha\varphi + \Delta_{\mathbb{H}}\varphi\right)d\eta dt + \mu\int_{\mathcal{H}_T}|u|^q\varphi Div_{\mathbb{H}}(a(\eta))d\eta dt\right\}, \end{aligned} \tag{3.11}$$

which implies

$$\begin{aligned}&\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt=\\&\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}\Big(Re(u)cos(\frac{\alpha\pi}{2})-Im(u)sin(\frac{\alpha\pi}{2})\Big)\,{}_t^C D_T^\alpha\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}Re(u)\Delta_{\mathbb{H}}\varphi d\eta dt\\&+\frac{\mu_1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}|u|^q\left(\varphi Div_{\mathbb{H}}(a(\eta))+a(\eta).\nabla_{\mathbb{H}}\varphi\right)d\eta dt.\end{aligned}\tag{3.12}$$

Then we find

$$\begin{aligned}&\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le\frac{2}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u||\,{}_t^C D_T^\alpha\varphi|d\eta dt\\&+\frac{1}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u||\Delta_{\mathbb{H}}\varphi|d\eta dt+\frac{|\mu_1|}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u|^q\left(|\varphi||Div_{\mathbb{H}}(a(\eta))|+|a(\eta)||\nabla_{\mathbb{H}}\varphi|\right)d\eta dt.\end{aligned}\tag{3.13}$$

By applying $\varepsilon-$ Young's inequality

$$ab\le\varepsilon a^p+C_\varepsilon b^{p'},p+p'=pp',a,b,\varepsilon,C_\varepsilon\ge0,$$

to the right-hand side of the above inequality, we obtain

$$\begin{aligned}&\left(1-\frac{\varepsilon(3-2|\mu_1|)}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le\\&\frac{C_\varepsilon}{|\lambda_1|}\left(2\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}d\eta dt\right)\\&+\frac{C_\varepsilon}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|\varphi|^{\frac{p}{p-q}}|Div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}d\eta dt\\&+\frac{C_\varepsilon}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}d\eta dt.\end{aligned}\tag{3.14}$$

Taking $\varepsilon=\frac{|\lambda_1|}{2(3-2|\mu_1|)}$, we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le\\&\frac{C}{|\lambda_1|}\left(2\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}d\eta dt\right)\\&+\frac{C}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|\varphi|^{\frac{p}{p-q}}|div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}d\eta dt\\&+\frac{C}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}d\eta dt.\end{aligned}\tag{3.15}$$

Now, we estimate each term of the right hand side of the above equality. By (3.8), we have

$$\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt=\left(\int_{\mathbb{H}}\varphi_2(\eta)d\eta\right)\left(\int_0^T\varphi_1^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi_1|^{\frac{p}{p-1}}dt\right).\tag{3.16}$$

On the other hand, from Lemma 2.2, we arrive at

$$\int_0^T\varphi_1^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi_1|^{\frac{p}{p-1}}dt=\left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\right]^{\frac{p}{p-1}}\frac{p-1}{p(m+1-\alpha)-(m+1)}T^{1-\frac{\alpha p}{p-1}}.\tag{3.17}$$

Therefore, by using the scaled variable

$$\tilde{\tau} = T^{-\alpha}\tau; \quad \tilde{x} = T^{-\frac{\alpha}{2}}x; \quad \tilde{y} = T^{-\frac{\alpha}{2}}y, \tag{3.18}$$

we obtain

$$\int_{\mathcal{H}_\mathcal{T}} \varphi^{-\frac{1}{p-1}} |\,{}_t^C D_T^\alpha \varphi|^{\frac{p}{p-1}} d\eta dt = C(m;p) T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha}, \tag{3.19}$$

where

$$C(m;p) = \left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\right]^{\frac{p}{p-1}} \frac{p-1}{p(m+1-\alpha)-(m+1)} \int_{0\le|\tilde{\eta}|_{\mathbb{H}}\le 2} \Phi^\omega(\tilde{\rho}) d\tilde{\rho}.$$

Similarly, we get

$$\int_{\mathcal{H}_\mathcal{T}} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} d\eta dt = \left(\int_0^T \varphi_1(t)dt\right)\left(\int_{\mathbb{H}} \varphi_2^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi_2|^{\frac{p}{p-1}} d\eta\right). \tag{3.20}$$

A simple computation yields

$$\int_0^T \varphi_1(t)dt = \frac{T}{m+1}. \tag{3.21}$$

Taking into account (3.10), we deduce

$$\int_{\mathcal{H}_\mathcal{T}} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} d\eta dt \le \frac{C}{m+1} T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha}. \tag{3.22}$$

From the condition (1.2) and using the same argument, we obtain

$$\int_{\mathcal{H}_\mathcal{T}} \varphi^{-\frac{q}{p-q}} |\varphi|^{\frac{p}{p-q}} |Div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}} d\eta dt \le C T^{1-\frac{\tau p}{p-q}+(N+1)\alpha}, \tag{3.23}$$

and

$$\int_{\mathcal{H}_\mathcal{T}} \varphi^{-\frac{q}{p-q}} |a(\eta)|^{\frac{p}{p-q}} |\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}} d\eta dt \le C T^{1-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+1)\alpha}. \tag{3.24}$$

Combining (3.15),(3.5),(3.22),(3.23) and (3.24), we get

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathcal{H}_\mathcal{T}} |u|^p \varphi d\eta dt + \frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta))\,{}_t^C D_T^\alpha \varphi d\eta dt \le \\ &\tfrac{C}{|\lambda_1|}\left(T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha} + |\mu_1| T^{1-\frac{\tau p}{p-q}+(N+1)\alpha} + |\mu_1| T^{1-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+1)\alpha}\right). \end{aligned} \tag{3.25}$$

Furthermore, it is not difficult to see that

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta))\,{}_t^C D_T^\alpha \varphi d\eta dt = \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)} T^{1-\alpha}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta))\Phi^\omega(\rho) d\eta dt. \tag{3.26}$$

Hence, we conclude

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathcal{H}_\mathcal{T}} |u|^p \varphi d\eta dt + \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta))\Phi^\omega(\rho) d\eta dt \le \\ &\tfrac{C}{|\lambda_1|}\left(T^{-\frac{\alpha p}{p-1}+(N+2)\alpha} + |\mu_1| T^{-\frac{\tau p}{p-q}+(N+2)\alpha} + |\mu_1| T^{-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+2)\alpha}\right). \end{aligned} \tag{3.27}$$

First, we suppose that

$$\lambda_1 \int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta)) d\eta > 0. \tag{3.28}$$

This implies that

$$\begin{aligned}&\tfrac{1}{\lambda_1}\tfrac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha;g(\eta))\Phi^\omega(\rho)d\eta dt \leq \\ &\tfrac{C}{|\lambda_1|}\left(T^{-\frac{\alpha p}{p-1}+(N+2)\alpha}+|\mu_1|T^{-\frac{\tau p}{p-q}+(N+2)\alpha}+|\mu_1|T^{-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+2)\alpha}\right).\end{aligned} \tag{3.29}$$

We have to discuss two cases:

Case 1: $\mu_1=0$ and $1<p<1+\frac{1}{N+1}$. In this case passing to the limit as $T\to+\infty$ in (3.29), we obtain

$$\frac{1}{\lambda_1}\int_{\mathbb{H}} G_1(\alpha;g(\eta))d\eta \leq 0,$$

which contradicts (3.28).

Case 2: $\mu_1\neq 0$ and $N<-2+p \quad min\left\{\frac{1}{p-1};\frac{\tau}{\alpha(p-q)};\frac{\alpha-2\nu}{2\alpha(p-q)}\right\}$. Passing to the limit as $T\to+\infty$ in (3.29), we get

$$\frac{1}{\lambda_1}\int_{\mathbb{H}} G_1(\alpha;g(\eta))d\eta \leq 0,$$

which contradicts (3.28). Next we suppose that

$$\lambda_2\int_{\mathcal{H}_\mathcal{T}} G_2(\alpha;g(\eta))d\eta > 0. \tag{3.30}$$

Observe that

$$v(\eta;t)=\frac{u(\eta,t)}{i},$$

is a global weak solution to the problem

$$i^\alpha\, {}_0^C D_t^\alpha v+\Delta_{\mathbb{H}}v=\lambda'|v|^p+\mu' a\left(\eta\right).\nabla_{\mathbb{H}}|v|^q,$$

$$v\left(\eta,0\right)=\tilde{g}\left(\eta\right),$$

where

$$\lambda'=\lambda_2+i(-\lambda_1)=\lambda_1'+i\lambda_2'; \qquad \mu'=\mu_2+i(-\lambda_1)=\mu_1'+i\mu_2',$$

and

$$\tilde{g}(\eta)=g_2(\eta)+i(-g_1(\eta))=\tilde{g}_1(\eta)+i\tilde{g}_2(\eta).$$

It can be easily seen that (3.30) is equivalent to

$$\lambda_1'\int_{\mathbb{H}} G_1(\alpha;\tilde{g}(\eta))d\eta > 0.$$

Therefore, from the previous case, if $\mu_2=0$ and $1<p<1+\frac{1}{N+1}$ we obtain the contradiction with (3.30). Similarly with the case $\mu_2\neq 0$ and $N<-2+p \quad min\left\{\frac{1}{p-1};\frac{\tau}{\alpha(p-q)};\frac{\alpha-2\nu}{2\alpha(p-q)}\right\}$, we get the contradiction with (3.30).

3.2. **Case of the system.**

Definition 3.2. *We say that the pair (u, v) is a local weak solution to (1.3) if the equalities*

$$\begin{aligned} &\lambda \int_{\mathcal{H}_T} |v|^p \varphi d\eta dt + i^\alpha \int_{\mathcal{H}_T} g(\eta) \, {}_t^C D_T^\alpha \varphi d\eta dt = \int_{\mathcal{H}_T} u \left(i^\alpha \, {}_t^C D_T^\alpha \varphi + \Delta_{\mathbb{H}} \varphi \right) d\eta dt \\ &+\mu \int_{\mathcal{H}_T} |v|^q \varphi \mathrm{div}_{\mathbb{H}}(a(\eta)) d\eta dt + \mu \int_{\mathcal{H}_T} |v|^q a(\eta).\nabla_{\mathbb{H}} \varphi d\eta dt, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} &\lambda \int_{\mathcal{H}_T} |u|^k \varphi d\eta dt + i^\beta \int_{\mathcal{H}_T} h(\eta) \, {}_t^C D_T^\beta \varphi d\eta dt = \int_{\mathcal{H}_T} v \left(i^\beta \, {}_t^C D_T^\beta \varphi + \Delta_{\mathbb{H}} \varphi \right) d\eta dt \\ &+\mu \int_{\mathcal{H}_T} |u|^\sigma \varphi \mathrm{div}_{\mathbb{H}}(b(\eta)) d\eta dt + \mu \int_{\mathcal{H}_T} |u|^q b(\eta).\nabla_{\mathbb{H}} \varphi d\eta dt, \end{aligned} \tag{3.32}$$

are satisfied for every test function $\varphi \in C_{t,\eta}^{1,2}(\mathcal{H}_T)$ with $\varphi(.,T) = 0$.
Moreover, if $T > 0$ can be arbitrarily chosen, then (u, v) is said to be a global weak solution to (1.3).

Our second main result is given by the following theorem:

Theorem 3.2. *Let $0 < \beta \leq \alpha < 1$; $p > q > 1$; $k > \sigma > 1$ and $g, h \in L^1(\mathbb{H})$. Suppose that one of the following cases holds:*

(I)

$$\alpha = \beta, \qquad \lambda_1 \int_{\mathbb{H}} \left[G_1(\alpha; g(\eta)) + G_2(\beta; h(\eta)) \right] d\eta > 0, \tag{3.33}$$

and

$$\mu_1 = 0, \qquad Q < min \left\{ \frac{1}{p-1}; \frac{1}{k-1} \right\},$$

or

$$\mu_1 \neq 0 \quad \text{and} \quad Q < min \left\{ \frac{\tau_1 p - \alpha(p-q)}{\alpha(p-q)}, \frac{\alpha q - \nu_1 p}{\alpha(p-q)}, \frac{\tau_2 k - \alpha(k-\sigma)}{\alpha(k-\sigma)}, \frac{\alpha\sigma - \nu_2 k}{\alpha(k-\sigma)} \right\}.$$

(II)

$$\beta < \alpha, \qquad \lambda_1 \int_{\mathbb{H}} G_1(\beta; h(\eta)) d\eta > 0, \tag{3.34}$$

and

$$\mu_1 = 0, \qquad Q < \frac{2}{\alpha+\beta} \left\{ \frac{\alpha}{k-1}, \frac{\beta p - \alpha(p-1)}{p-1} \right\},$$

or $\mu_1 \neq 0$ and

$$Q < \frac{2}{\alpha+\beta}$$

$$min \left\{ \frac{\tau_1 p - \alpha(p-q)}{p-q}, \frac{2\alpha q - (\alpha+\beta+2\nu_1)p}{2(p-q)}, \frac{\tau_2 k - \alpha(k-\sigma)}{k-\sigma}, \frac{(\alpha+\beta)k - 2\alpha(k-\sigma) - 2\nu_2 k}{2(k-\sigma)} \right\}.$$

(III)

$$\alpha=\beta, \qquad \lambda_2\int_{\mathbb{H}}\left[G_1(\alpha;g(\eta))+G_2(\beta;h(\eta))\right]d\eta>0, \tag{3.35}$$

and

$$\mu_2=0, \qquad Q<min\left\{\frac{1}{p-1}, \quad \frac{1}{k-1}\right\},$$

or

$$\mu_2\neq 0, \qquad Q<min\left\{\frac{\tau_1 p-\alpha(p-q)}{\alpha(p-q)},\frac{\alpha q-\nu_1 p}{\alpha(p-q)},\frac{\tau_2 k-\alpha(k-\sigma)}{\alpha(k-\sigma)},\frac{\alpha\sigma-\nu_2 k}{\alpha(k-\sigma)}\right\}.$$

(IV)

$$\alpha<\beta, \qquad \lambda_2\int_{\mathbb{H}}G_1(\beta;h(\eta))d\eta>0, \tag{3.36}$$

and

$$\mu_2=0, \qquad Q<\frac{2}{\alpha+\beta}\quad min\left\{\frac{\alpha}{k-1},\frac{\beta p-\alpha(p-1)}{p-1}\right\},$$

or

$\mu_2\neq 0$ *and*

$$Q<\frac{2}{\alpha+\beta}$$

$$min\left\{\frac{\tau_1 p-\alpha(p-q)}{p-q},\frac{2\alpha q-(\alpha+\beta+2\nu_1)p}{2(p-q)},\frac{\tau_2 k-\alpha(k-\sigma)}{k-\sigma},\frac{(\alpha+\beta)k-2\alpha(k-\sigma)-2\nu_2 k}{2(k-\sigma)}\right\}.$$

Then the system (1.3) admits no global weak solution.

Proof. Let φ be the test function defined by (3.8) where φ_2 is given by

$$\varphi_2(\eta)=\Phi^{\omega}\left(\frac{\tau^2+|x|^4+|y|^4}{T^{2(\alpha+\beta)}}\right),$$

where Φ is given by formula(3.9).

Suppose that (u,v) is a global weak solution to (1.3). By using the definition of weak solution, we get

$$\begin{aligned}&Re\left(\lambda\int_{\mathcal{H}_T}|v|^p\varphi d\eta dt+i^{\alpha}\int_{\mathcal{H}_T}g(\eta)\ {}^{C}_{t}D^{\alpha}_{T}\varphi d\eta dt\right)=Re\left(\mu\int_{\mathcal{H}_T}|v|^q a(\eta).\nabla_{\mathbb{H}}\varphi d\eta dt\right)\\&Re\left(\int_{\mathcal{H}_T}u\left(i^{\alpha}\ {}^{C}_{t}D^{\alpha}_{T}\varphi+\Delta_{\mathbb{H}}\varphi\right)d\eta dt\right)+Re\left(\mu\int_{\mathcal{H}_T}|v|^q\varphi div_{\mathbb{H}}(a(\eta))d\eta dt\right).\end{aligned} \tag{3.37}$$

First we repeat the same calculation as above, we obtain

$$\begin{aligned}&\int_{\mathcal{H}_T}|v|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_T}G_1(\alpha;g(\eta))\ {}^{C}_{t}D^{\alpha}_{T}\varphi d\eta dt\le\frac{2}{|\lambda_1|}\int_{\mathcal{H}_T}|u||\ {}^{C}_{t}D^{\alpha}_{T}\varphi|d\eta dt\\&+\frac{1}{|\lambda_1|}\int_{\mathcal{H}_T}|u||\Delta_{\mathbb{H}}\varphi|d\eta dt+\frac{|\mu_1|}{|\lambda_1|}\int_{\mathcal{H}_T}|v|^q\left(|\varphi||Div_{\mathbb{H}}(a(\eta))|+|a(\eta)||\nabla_{\mathbb{H}}\varphi|\right)d\eta dt.\end{aligned} \tag{3.38}$$

Also, using the arguments of the previous theorem, we arrive at

$$\begin{aligned}&\left(1-2\tfrac{\varepsilon|\mu_1|}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|v|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}^C_t D^\alpha_T\varphi d\eta dt\leq\\&\tfrac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{k-1}}\left(|\ {}^C_t D^\alpha_T\varphi|^{\frac{k}{k-1}}+|\Delta_{\mathbb{H}}\varphi|^{\frac{k}{k-1}}\right)d\eta dt+3\frac{\varepsilon}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u|^k\varphi d\eta dt\right.\\&\left.+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}\left(|\varphi|^{\frac{p}{p-q}}|div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}+|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}\right)d\eta dt\right).\end{aligned}\tag{3.39}$$

Similarly, we have

$$\begin{aligned}&\left(1-2\tfrac{\varepsilon|\mu_1|}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|u|^k\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}^C_t D^\beta_T\varphi d\eta dt\leq\\&\tfrac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}\left(|\ {}^C_t D^\beta_T\varphi|^{\frac{p}{p-1}}+|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}\right)d\eta dt+3\frac{\varepsilon}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|v|^p\varphi d\eta dt.\right.\\&\left.+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{\sigma}{k-\sigma}}\left(|\varphi|^{\frac{k}{k-\sigma}}|Div_{\mathbb{H}}(b(\eta))|^{\frac{k}{k-\sigma}}+|b(\eta)|^{\frac{k}{k-\sigma}}|\nabla_{\mathbb{H}}\varphi|^{\frac{k}{k-\sigma}}d\eta dt\right)\right.\end{aligned}\tag{3.40}$$

Next, adding (3.39) to (3.40) and taking $\varepsilon=\frac{|\lambda_1|}{2(2|\mu_1|-3|)}$, we get

$$\begin{aligned}&\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}^C_t D^\alpha_T\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}^C_t D^\beta_T\varphi d\eta dt\\&+\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}\left(|v|^p+|u|^k\right)\varphi d\eta dt\leq\frac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{k-1}}\left(|\ {}^C_t D^\alpha_T\varphi|^{\frac{k}{k-1}}+|\Delta_{\mathbb{H}}\varphi|^{\frac{k}{k-1}}\right)\right)d\eta dt\\&+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}\left(|\varphi|^{\frac{p}{p-q}}|Div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}+|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}\right)d\eta dt\\&+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}\left(|\ {}^C_t D^\beta_T\varphi|^{\frac{p}{p-1}}+|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}\right)d\eta dt\\&+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{\sigma}{k-\sigma}}\left(|\varphi|^{\frac{k}{k-\sigma}}|Div_{\mathbb{H}}(b(\eta))|^{\frac{k}{k-\sigma}}|b(\eta)|^{\frac{k}{k-\sigma}}|\nabla_{\mathbb{H}}\varphi|^{\frac{k}{k-\sigma}}d\eta dt\right).\end{aligned}\tag{3.41}$$

At this stage, we use the scaled variable

$$\tilde{\tau}=T^{-(\alpha+\beta)}\tau,\quad \tilde{x}=T^{-\frac{(\alpha+\beta)}{2}}x,\quad \tilde{y}=T^{-\frac{(\alpha+\beta)}{2}}y,\tag{3.42}$$

to obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}\left(|v|^p+|u|^k\right)\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}^C_t D^\alpha_T\varphi d\eta dt\\&+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}^C_t D^\beta_T\varphi d\eta dt\leq C\left(T^{1-\frac{2k}{k-1}+(N+1)(\alpha+\beta)}+T^{1-\frac{\alpha k}{k-1}+(N+1)(\alpha+\beta)}\right)\\&+C\left(T^{1-\frac{2p}{p-1}+(N+1)(\alpha+\beta)}+T^{1-\frac{\beta p}{p-1}+(N+1)(\alpha+\beta)}\right)\\&+|\mu_1|\left(T^{1-\frac{\tau_1 p}{p-q}+(N+1)(\alpha+\beta)}+T^{1-\frac{\theta p}{p-q}+\frac{\nu_1 p}{p-q}+(N+1)(\alpha+\beta)}\right)\\&+|\mu_1|\left(T^{1-\frac{\tau_2 k}{k-\sigma}+(N+1)(\alpha+\beta)}+T^{1-\frac{\theta k}{k-\sigma}+\frac{\nu_2 k}{k-\sigma}+(N+1)(\alpha+\beta)}\right).\end{aligned}\tag{3.43}$$

Furthermore, we have

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}^C_t D^\alpha_T\varphi d\eta dt=\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}T^{1-\alpha}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\Phi^\omega(\rho)d\eta dt,\tag{3.44}$$

and

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta,h(\eta))\ {}_t^C\ D_T^\beta\varphi d\eta dt = \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{1-\beta}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta;h(\eta))\Phi^\omega(\rho)d\eta dt. \quad (3.45)$$

Therefore, we get

$$\begin{aligned}&\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha;g(\eta))\Phi^\omega(\rho)d\eta dt\\&+\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{\alpha-\beta}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta;h(\eta))\Phi^\omega(\rho)d\eta dt\\&\leq C\left(T^{\theta_1}+T^{\theta_2}+T^{\theta_3}+T^{\theta_4}\right)+C|\mu_1|\left(T^{\theta_5}+T^{\theta_6}+T^{\theta_7}+T^{\theta_8}\right),\end{aligned} \quad (3.46)$$

where

$$\theta_1=\alpha-\frac{2k}{k-1}+(N+1)(\alpha+\beta), \qquad \theta_2=\alpha-\frac{\alpha k}{k-1}+(N+1)(\alpha+\beta),$$

$$\theta_3=\alpha-\frac{2p}{p-1}+(N+1)(\alpha+\beta), \qquad \theta_4=\alpha-\frac{\beta p}{p-1}+(N+1)(\alpha+\beta),$$

$$\theta_5=\alpha-\frac{\tau_1 p}{p-q}+(N+1)(\alpha+\beta), \qquad \theta_6=\alpha-\frac{(\alpha+\beta)p}{2(p-q)}+\frac{\tau_1 p}{p-q}+(N+1)(\alpha+\beta),$$

$$\theta_7=\alpha-\frac{\tau_2 k}{k-\sigma}+(N+1)(\alpha+\beta), \qquad \theta_8=\alpha-\frac{(\alpha+\beta)k}{2(k-\sigma)}+\frac{\tau_2 k}{k-\sigma}+(N+1)(\alpha+\beta).$$

Suppose now that

$$\alpha=\beta \quad \text{and} \qquad \lambda_1\int_{\mathbb{H}}\left[G_1(\alpha;g(\eta))+G_2(\beta;h(\eta))\right]d\eta>0. \quad (3.47)$$

We distinguish two cases:

Case 1: $\mu_1=0$, $Q<min\{\frac{1}{p-1};\frac{1}{k-1}\}$. In this case, passing to the limit as $T\longrightarrow+\infty$ in (3.46), we obtain a contradiction with (3.47).

Case 2: $\mu_1\neq 0$; $Q<min\left\{\frac{\tau_1 p-\alpha(p-q)}{\alpha(p-q)};\frac{\alpha q-\nu_1 p}{\alpha(p-q)};\frac{\tau_2 k-\alpha(k-\sigma)}{\alpha(k-\sigma)};\frac{\alpha\sigma-\nu_2 k}{\alpha(k-\sigma)}\right\}$. Similarly, passing to the limit as $T\longrightarrow+\infty$ in (3.46), we obtain a contradiction with (3.47).

Suppose now that

$$\beta<\alpha; \qquad \lambda_1\int_{\mathbb{H}} G_1(\beta;h(\eta))d\eta>0. \quad (3.48)$$

We have to distinguish to cases:

Case 1: $\mu_1=0$, $Q<\frac{2}{\alpha+\beta}\quad min\left\{\frac{\alpha}{k-1};\frac{\beta p-\alpha(p-1)}{p-1}\right\}$. In this case, passing to the limit as $T\longrightarrow+\infty$ in (3.46), we get a contradiction with (3.48).

Case 2: $\mu_1\neq 0, Q<\frac{2}{\alpha+\beta}\quad min\left\{\frac{\tau_1 p-\alpha(p-q)}{p-q};\frac{2\alpha q-(\alpha+\beta+2\nu_1)p}{2(p-q)};\frac{\tau_2 k-\alpha(k-\sigma)}{k-\sigma};\frac{(\alpha+\beta)k-2\alpha(k-\sigma)-2\nu_2 k}{2(k-\sigma)}\right\}$.

Similarly, passing to the limit as $T \longrightarrow +\infty$ in (3.46), we obtain a contradiction with (3.48). Next, we consider the case $\lambda_2 \neq 0$. Observe that

$$(U(\eta;t),\ V(\eta;t)) = \left(\frac{u(\eta;t)}{i},\ \frac{v(\eta;t)}{i}\right),$$

is a global weak solution to the system

$$\begin{aligned} i^{\alpha}\, {}_0^C D_t^{\alpha} U + \Delta_{\mathbb{H}} U &= \lambda' |V|^p + \mu' a(\eta) .\nabla_{\mathbb{H}} |V|^q, \\ i^{\beta}\, {}_0^C D_t^{\beta} V + \Delta_{\mathbb{H}} V &= \lambda' |U|^k + \mu' b(\eta) .\nabla_{\mathbb{H}} |U|^{\sigma}, \\ U(\eta,0) = \tilde{g}(\eta); &\qquad V(\eta,0) = \tilde{h}(\eta), \end{aligned}$$

where

$$\lambda' = \lambda_2 + i(-\lambda_1) = \lambda_1' + i\lambda_2', \mu' = \mu_2 + i(-\lambda_1) = \mu_1' + i\mu_2',$$

$$\tilde{g}(\eta) = g_2(\eta) + i(-g_1(\eta)) = \tilde{g}_1(\eta) + i\tilde{g}_2(\eta), \tilde{h}(\eta) = h_2(\eta) + i(-h_1(\eta)) = \tilde{h}_1(\eta) + i\tilde{h}_2(\eta).$$

Therefore, from the previous study, if one of the cases **(III)** or **(IV)** holds, we obtain a contradiction.

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International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:50-67

www.journalmim.com

ON THE GEOMETRY OF CONFORMAL ANTI-INVARIANT $\xi^{\perp}-$SUBMERSIONS

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Abstract. Lee [Anti-invariant $\xi^{\perp}-$ Riemannian submersions from almost contact manifolds, Hacettepe Journal of Mathematics and Statistic, 42(3), (2013), 231-241.] defined and studied anti-invariant $\xi^{\perp}-$Riemannian submersions from almost contact manifolds. The main goal of this paper is to consider conformal anti-invariant $\xi^{\perp}-$submersions (it means the Reeb vector field ξ is a horizontal vector field) from almost contact metric manifolds onto Riemannian manifolds as a generalization of anti-invariant $\xi^{\perp}-$Riemannian submersions. More precisely, we obtain the geometries of the leaves of $\ker\pi_*$ and $(\ker\pi_*)^{\perp}$, including the integrability of the distributions, the geometry of foliations, some conditions related to totally geodesicness and harmonicty of the submersions. Finally, we show that there are certain product structures on the total space of a conformal anti-invariant $\xi^{\perp}-$submersion.

1. Introduction

In the 1960s, B. O'Neill [22] and A. Gray [16] independently studied the notion of Riemannian submersions between Riemannian manifolds. In [31], B. Watson defined almost Hermitian submersions, meaning submersions defined on the Riemannian submersions between almost Hermitian manifolds. The author showed that the Riemannian submersion is also an almost complex mapping and consequently the horizontal and vertical distributions

Received:2018-02-10 *Revised: 2018-03-15* *Accepted:2018-03-19*

2010 Mathematics Subject Classification: 53C15, 53C40.

Key words: Almost contact metric manifold, conformal submersion, anti-invariant $\xi^{\perp}-$Riemannian submersion, conformal anti-invariant $\xi^{\perp}-$submersion

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are invariant with respect to the tensor field of type $(1,1)$ of the total space.

In [28], B. Sahin defined anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. We refer to some papers ([5], [15], [24], [26], [29]) related to the notion and the book [30]. The notion of almost contact Riemannian submersions between almost contact metric manifolds initiated by Chinea in [10]. He obtained the differential geometric properties among total space, fibers and base spaces.

One the other hand, Fuglede [11] and Ishihara [17], as a generalization of Riemannian submersion, introduced independently horizontally conformal submersions (see also: [4], [13], [23]). Gudmundsson and Wood [14], as a generalization of holomorphic submersions, defined the notion of conformal holomorphic submersions and obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism (see also [7], [8] and [9]). Recently, the first author of that paper in [1] considered conformal anti-invariant submersions, meaning submersions defined on cosymplectic manifolds such that the vertical distribution is anti invariant with respect to the almost contact structure (see also: [18], [19]). In this paper, we consider conformal anti-invariant $\xi^{\perp}$−submersions from an almost contact metric manifold under the assumption that the fibers are anti-invariant with respect to the tensor field of type $(1,1)$ of the almost contact manifold.

The paper is organized as follows. Section 2, we give some basic notions related to almost contact metric manifolds and conformal submersions. In third section, we introduce conformal anti-invariant $\xi^{\perp}$−submersions from almost contact metric manifolds onto Riemannian manifolds, and give main results for the geometry of a conformal anti-invariant $\xi^{\perp}$−submersion. The last section, we show that there are certain product structures on the total space of a conformal anti-invariant $\xi^{\perp}$−submersion.

2. Preliminaries

In this paper, all manifolds, vector fields and maps are assumed to be smooth unless otherwise stated.

2.1. **Almost contact metric manifolds.** Let (M, g_M) be an almost contact metric manifold with structure tensors (ϕ, ξ, η, g_M) where ϕ is a tensor field of type (1,1), ξ is a Reeb vector field, η is a 1-form and g_M is the Riemannian metric on M. Then these tensors satisfy [3]

$$\phi\xi = 0, \quad \eta o\phi = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

$$\phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad g_M(\phi X, \phi Y) = g_M(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

where I denotes the identity endomorphism of TM and X, Y are any vector fields on M. Moreover, if M is Sasakian [27], then we have

$$(\nabla_X \phi)Y = -g_M(X, Y)\xi + \eta(Y)X \quad \text{and} \quad \nabla_X \xi = \phi X, \tag{2.3}$$

where ∇ is the connection of Levi-Civita covariant differentiation.

2.2. **Conformal submersions.** Let $\varphi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a smooth map between Riemannian manifolds, and let $q \in M$. Then φ is called horizontally weakly conformal or semi conformal at q [4] if either (i) $d\varphi_q = 0$, or (ii) $d\varphi_q$ maps horizontal space $\mathcal{H}_q = (ker(d\varphi_q))^{\perp}$ conformally onto $T_{\varphi_*}N$, i.e., $d\varphi_q$ is surjective and there exists a number $\Lambda(q) \neq 0$ such that

$$g_N(d\varphi_q X, d\varphi_q Y) = \Lambda(q) g_M(X, Y) \ (X, Y \in \mathcal{H}_q).$$

We call the point q is of type (i) as a critical point if it satisfies the type (i), and we shall call the point q a regular point if it satisfied the type (ii). At a critical point, $d\varphi_q$ has rank 0; at a regular point, $d\varphi_q$ has rank n and φ is submersion. Also, the number $\Lambda(q)$ is called the *square dilation* (of φ at q). The map φ is called *horizontally weakly conformal* or *semi conformal* (on M) if it is horizontally weakly conformal at every point of M and it has no critical point, then we call it a (*horizontally conformal submersion*).

A vector field $Z \in \Gamma(TM)$ is called a basic vector field if $Z \in \Gamma((ker\pi_*)^{\perp})$ and $\pi-$related with a vector field $\bar{Z} \in \Gamma(TN)$ which means that $(\pi_{*q} Z_q) = \bar{Z}(\pi(q)) \in \Gamma(TN)$ for any $q \in \Gamma(TM)$.

O'Neill's tensors T and A defined for any $E, F \in \Gamma(TM)$ as follows;

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.4}$$

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.5}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections (see [12]). And also, by using (2.4) and (2.5), for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$, we have

$$\nabla_V W = T_V W + \hat{\nabla}_V W \tag{2.6}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + T_V X \tag{2.7}$$

$$\nabla_X V = A_X V + \mathcal{V}\nabla_X V \tag{2.8}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y \tag{2.9}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = A_X V$. Then it is well-know that

$$-g(A_X E, F) = g(E, A_X F) \text{ and } -g(T_V E, F) = g(E, T_V F)$$

for all $E, F \in T_x M$. T is exactly the second fundamental form of the fibres of π. For the special case when π is horizontally conformal we have the following:

Proposition 2.1. *([13]) Let $\pi : (M^m, g) \longrightarrow (N^n, h)$ be a horizontally conformal submersion with dilation λ and X, Y be horizontal vectors, then*

$$A_X Y = \frac{1}{2}\{\mathcal{V}[X, Y] - \lambda^2 g(X, Y) grad_{\mathcal{V}}(\frac{1}{\lambda^2})\}. \tag{2.10}$$

Definition 2.1. *Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\pi : M \longrightarrow N$ is a smooth map between them. The second fundamental form of π is given by*

$$(\nabla\pi_*)(X, Y) = \nabla^{\pi}_X \pi_*(Y) - \pi_*(\nabla^M_X Y) \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$, where ∇^{π} is the pullback connection. It is obvious that the second fundamental form $(\nabla\pi_*)$ is symmetric.

Lemma 2.1. [32] *Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \longrightarrow N$ is a smooth map between them. Then we have*

$$\nabla^{\varphi}_X \varphi_*(Y) - \nabla^{\varphi}_Y \varphi_*(X) - \varphi_*([X, Y]) = 0$$

for $X, Y \in \Gamma(TM)$.

Remark 2.1. *From Lemma 2.1, for any X is basic vector field and $Y \in \Gamma(ker\pi_*)$, we obtain $[X, V] \in \Gamma(ker\pi_*)$. So, in this paper we assume that all horizontal vector fields are basic vector fields.*

Recall that π is called harmonic if the tension field $\tau(\pi) = trace(\nabla\pi_*) = 0$. (for details, see [4]).

Lemma 2.2. [4] *Let $\pi : M \longrightarrow N$ be a horizontally conformal submersion. Then, we have*

(a) $(\nabla\pi_*)(X, Y) = X(\ln\lambda)\pi_* Y + Y(\ln\lambda)\pi_* X - g(X, Y)\pi_*(\nabla\ln\lambda)$;

(b) $(\nabla\pi_*)(V, W) = -\pi_*(T_V W)$;

(c) $(\nabla\pi_*)(X, V) = -\pi_*(\nabla^M_X V) = -\pi_*(A_X V)$

for any $V \in \Gamma(ker\pi_*)$ and $X, Y \in (ker\pi_*)^{\perp}$.

Finally, we will mention the following from [25].

Let g_1 be a Riemannian metric tensor on the manifold $N = M_1 \times M_2$ and assume that the canonical foliations $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ intersect perpendiculary everywhere. Then g is the metric tensor of a usual product of Riemannian manifolds $\Longleftrightarrow$ $\mathcal{D}_{M_1}$ and $\mathcal{D}_{M_2}$ are totally geodesic foliations.

3. Conformal anti-invariant $\xi^{\perp}-$submersions

In this section, we first define conformal anti-invariant $\xi^{\perp}-$submersions from an almost contact metric manifold onto a Riemannian manifold and derive the integrability of distributions, the geometry of foliations, some conditions related to totally geodesicness and harmonicity of the map. First of all, we give the definition of the submersion as follows:

Definition 3.1. *Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) be a Riemannian manifold. We suppose that there exist a horizontally conformal submersion $\pi : M \longrightarrow N$ such that ξ is normal to $ker\pi_*$ and $ker\pi_*$ is anti-invariant with respect to ϕ, i.e., $\phi(ker\pi_*) \subset (ker\pi_*)^{\perp}$. Then we say that π is a conformal anti-invariant $\xi^{\perp}-$submersion.*

Assume that if $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ is a conformal anti-invariant $\xi^{\perp}-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then from Definition 3.1, we have $\phi(ker\pi_*)^{\perp} \cap ker\pi_* \neq 0$. We denote the complementary orthogonal distribution to $\phi(ker\pi_*)$ in $(ker\pi_*)^{\perp}$ by μ. Then we write

$$(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus \mu. \tag{3.12}$$

Here, μ is an invariant distribution of $(ker\pi_*)^{\perp}$, with respect to ϕ and contains ξ. Given $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{3.13}$$

where $\mathcal{B}X \in \Gamma(ker\pi_*)$ and $\mathcal{C}X \in \Gamma(\mu)$. On the other hand, since $\pi_*((ker\pi_*)^{\perp}) = TN$ and π is a conformal submersion, using (3.13) we obtain $\lambda^{-2} g_N(\pi_*\phi V, \pi_*\mathcal{C}X) = 0$ for any $X \in \Gamma((ker\pi_*)^{\perp})$ and $V \in \Gamma(ker\pi_*)$, which implies that

$$TN = \pi_*(\phi ker\pi_*) \oplus \pi_*(\mu). \tag{3.14}$$

Remark 3.1. *We note that every anti-invariant $\xi^{\perp}-$submersion from an almost contact manifold onto a Riemannian manifold is a conformal anti-invariant $\xi^{\perp}-$submersion with $\lambda = 1$ [20].*

Lemma 3.1. *Let π be a conformal anti-invariant $\xi^{\perp}$-submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$A_X\xi = -\mathcal{B}X, \tag{3.15}$$

$$T_V\xi = 0, \tag{3.16}$$

$$g_M(\mathcal{C}Y, \phi W) = 0, \tag{3.17}$$

$$g_M(\nabla^M_X \mathcal{C}Y, \phi W) = -g_M(\mathcal{C}Y, \phi A_X W) \tag{3.18}$$

for $Y, \xi, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. By using (2.3), (2.9) and (3.13) we have (3.15). Using (2.3) and (2.7) we get (3.16). Given $Y \in \Gamma((ker\pi_*)^{\perp})$, $W \in \Gamma(ker\pi_*)$ and using (2.2), we have

$$g_M(\mathcal{C}Y, \phi W) = g_M(\phi Y - \mathcal{B}Y, \phi W) = g_M(\phi Y, \phi W) = g_M(Y, W) + \eta(Y)\eta(W) = g_M(Y, W) = 0,$$

due to $\mathcal{B}Y \in \Gamma(ker\pi_*)$ and $\phi W, \xi \in \Gamma((ker\pi_*)^{\perp})$. Differentiating (3.17) with respect to X, we get

$$\begin{aligned} g_M(\nabla^M_X \mathcal{C}Y, \phi W) &= -g_M(\mathcal{C}Y, \nabla^M_X \phi W) \\ &= -g_M(\mathcal{C}Y, (\nabla^M_X \phi)W) - g_M(\mathcal{C}Y, \phi(\nabla^M_X W)) \\ &= -g_M(\mathcal{C}Y, \phi(\nabla^M_X W)) \\ &= -g_M(\mathcal{C}Y, \phi A_X W) - g_M(\mathcal{C}Y, \phi \mathcal{V}\nabla^M_X W) \\ &= -g_M(\mathcal{C}Y, \phi A_X W) \end{aligned}$$

due to $\phi\mathcal{V}\nabla^M_X W \in \Gamma(\phi ker\pi_*)$. One can easily obtain the others.

As we know the distribution $ker\pi_*$ is integrable, we only deal with the integrability of the distribution $(ker\pi_*)^{\perp}$ and the geometry of the distributions.

Theorem 3.1. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$–submersion. Then the following conditions are equivalent to each other;*

(a) *The distribution $(ker\pi_*)^{\perp}$ is integrable,*

(b) $$\begin{aligned} \lambda^{-2} g_N(\nabla^N_{\pi_* Y}\pi_*\mathcal{C}X - \nabla^{\pi}_X \pi_*\mathcal{C}Y, \pi_*\phi W) &= g_M(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y \\ &\quad - 2g_M(\mathcal{C}X, Y)\ln\lambda - \eta(Y)X + \eta(X)Y, \phi W) \end{aligned}$$

for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. In view of (2.2) and (2.3), we get

$$g_M(\nabla^M_X Y, W) = g_M(\nabla^M_X \phi Y, \phi W) - \eta(Y) g_M(X, \phi W) \tag{3.19}$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $W \in \Gamma(ker\pi_*)$. Then, using (3.13) and (3.19), we find

$$\begin{aligned} g_M([X,Y],W) &= g_M(\nabla^M_X \phi Y, \phi W) - g_M(\nabla^M_Y \phi X, \phi W) - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W) \\ &= g_M(\nabla^M_X \mathcal{B}Y, \phi W) + g_M(\nabla^M_X \mathcal{C}Y, \phi W) - g_M(\nabla^M_Y \mathcal{B}X, \phi W) - g_M(\nabla^M_Y \mathcal{C}X, \phi W) \\ &\quad - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Using the property of π and (2.8) we derive

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X, \phi W) + \lambda^{-2} g_N(\pi_*(\nabla^M_X \mathcal{C}Y), \pi_*\phi W) \\ &\quad - \lambda^{-2} g_N(\pi_*(\nabla^M_Y \mathcal{C}X), \pi_*\phi W) - \eta(Y) g_M(X, \phi V) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Hence, from (2.11) and Lemma 2.2 we get

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X, \phi W) - g_M(\mathcal{H}\nabla \ln\lambda, X) g_M(\mathcal{C}Y, \phi W) \\ &\quad - g_M(\mathcal{H}\nabla \ln\lambda, \mathcal{C}Y) g_M(X, \phi W) + g_M(X, \mathcal{C}Y) g_M(\mathcal{H}\nabla \ln\lambda, \phi W) \\ &\quad + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_*\phi W) + g_M(\mathcal{H}\nabla \ln\lambda, Y) g_M(\mathcal{C}X, \phi W) \\ &\quad + g_M(\mathcal{H}\nabla \ln\lambda, \mathcal{C}X) g_M(Y, \phi W) - g_M(Y, \mathcal{C}X) g_M(\mathcal{H}\nabla \ln\lambda, \phi W) \\ &\quad - \lambda^{-2} g_N(\nabla^N_{\pi_* Y} \pi_* \mathcal{C}X, \pi_*\phi W) - \eta(Y) g_M(X, \phi W) + \eta(X) g_M(Y, \phi W). \end{aligned}$$

Furthermore, by using (3.17), we obtain

$$\begin{aligned} g_M([X,Y],W) &= g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y - 2g_M(\mathcal{C}X, Y)\ln\lambda \\ &\quad - \eta(Y)X + \eta(X)Y, \phi W) - \lambda^{-2} g_N(\nabla^N_{\pi_* Y} \pi_* \mathcal{C}X - \nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_*\phi W). \end{aligned}$$

which means that $(a) \Leftrightarrow (b)$.

Using the integrability of $(ker\pi_*)^\perp$, from Theorem 3.1, we deduce:

Theorem 3.2. *Let π be a conformal anti-invariant $\xi^\perp-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two assertions below imply the third;*

(a) *The distribution $(ker\pi_*)^\perp$ is integrable.*

(b) *The map π is horizontally homothetic submersion.*

(c) $g_N(\nabla^\pi_Y \pi_* \mathcal{C}X - \nabla^\pi_X \pi_* \mathcal{C}Y, \pi_*\phi W) = \lambda^2 g_M(A_X \mathcal{B}Y - A_Y \mathcal{B}X - \eta(Y)X + \eta(X)Y, \phi W)$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $W \in \Gamma(ker\pi_*)$.

Proof. By the proof of Theorem 3.1, for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, we have

$$g_M([X,Y],W) = g_M(A_X\mathcal{B}Y - A_Y\mathcal{B}X - \mathcal{C}Y(\ln\lambda)X + \mathcal{C}X(\ln\lambda)Y - 2g_M(\mathcal{C}X,Y)\ln\lambda$$
$$- \eta(Y)X + \eta(X)Y, \phi W) - \lambda^{-2}g_N(\nabla^{\pi}_{Y}\pi_*\mathcal{C}X - \nabla^{\pi}_{X}\pi_*\mathcal{C}Y, \pi_*\phi W).$$

Now, if we have (a) and (c), then we obtain

$$\begin{aligned}&- g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(X,\phi W) + g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}X)g_M(Y,\phi W)\\ &- 2g_M(\mathcal{C}X,Y)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0.\end{aligned} \tag{3.20}$$

Now, taking $Y = \phi W$ in (3.20) for $W \in \Gamma(ker\pi_*)$, using (2.2) and (3.17), we derive

$$\begin{aligned}g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}X)g_M(\phi W,\phi W)) &= g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}X)\{g_M(W,W) - \eta(W)\eta(W)\}\\ &= g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}X)g_M(W,W) = 0.\end{aligned}$$

Hence λ is a constant on $\Gamma(\mu)$. On the other hand, taking $Y = \mathcal{C}X$ in (3.20) for $X \in \Gamma(\mu)$ and using (3.17) we obtain

$$\begin{aligned}&- g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}^2Y)g_M(X,\phi W) + g_M(\mathcal{H}\nabla\ln\lambda,\mathcal{C}X)g_M(\mathcal{C}X,\phi W)\\ &- 2g_M(\mathcal{C}X,\mathcal{C}X)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0.\end{aligned}$$

Thus, we get $2g_M(\mathcal{C}X,\mathcal{C}X)g_M(\mathcal{H}\nabla\ln\lambda,\phi W) = 0$ which means that the dilation λ is a constant on $\Gamma(\phi ker\pi_*)$. One can easily obtain the others.

Remark 3.2. *We assume that $(ker\pi_*)^{\perp} = \phi ker\pi_* \oplus \{\xi\}$. Using (3.13) one can prove that $\mathcal{C}X = 0$.*

Hence we obtain,

Corollary 3.1. *Let $\pi : (M,\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*)\oplus <\xi>$. Then the following conditions are equivalent to each other;*

(a) *The distribution $(ker\pi_*)^{\perp}$ is integrable*

(b) $A_X\phi Y + \eta(X)Y = A_Y\phi X + \eta(Y)X$

(c) $(\nabla\pi_*)(X,\phi Y) + \eta(Y)\pi_*X = (\nabla\pi_*)(Y,\phi X) + \eta(X)\pi_*Y$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$.

For the geometry of the distribution $(ker\pi_*)^{\perp}$, we get:

Theorem 3.3. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion. Then the following assertions are equivalent to each other;*

(a) *$(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*

(b) $-\lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi W) = g_M(A_X BY - \mathcal{C}Y(\ln \lambda)X + g_M(X, \mathcal{C}Y) \ln \lambda - \eta(Y)X, \phi W)$

for $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$, $W \in \Gamma(ker\pi_*)$ and by using (2.2), (2.8), (2.9), (3.12), (3.13) and (3.19), we have

$$g_M(\nabla^M_X Y, W) = g_M(A_X BY, \phi W) + g_M(\nabla^M_X \mathcal{C}Y, \phi W) - \eta(Y) g_M(X, \phi W).$$

Using the property of π, (2.11) and Lemma (2.2) we arrive at

$$\begin{aligned} g_M(\nabla^M_X Y, W) &= g_M(A_X BY, \phi W) - \lambda^{-2} g_M(\mathcal{H}\nabla \ln \lambda, X) g_N(\pi_* \mathcal{C}Y, \pi_* \phi W) \\ &\quad - \lambda^{-2} g_M(\mathcal{H}\nabla \ln \lambda, \mathcal{C}Y) g_N(\pi_* X, \pi_* \phi W) \\ &\quad + \lambda^{-2} g_M(X, \mathcal{C}Y) g_N(\pi_*(\mathcal{H}\nabla \ln \lambda, \pi_* \phi W) \\ &\quad + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi V) - \eta(Y) g_M(X, \phi W) \end{aligned}$$

and using Definition 3.1 and (3.17) we arrive at

$$\begin{aligned} g_M(\nabla^M_X Y, W) &= g_M(A_X BY - \mathcal{C}Y(\ln \lambda)X + g_M(X, \mathcal{C}Y) \ln \lambda - \eta(Y)X, \phi W) \\ &\quad + \lambda^{-2} g_N(\nabla^N_{\pi_* X} \pi_* \mathcal{C}Y, \pi_* \phi W) \end{aligned}$$

which tells that $(i) \Leftrightarrow (ii)$.

From Theorem 3.3, we obtain

Theorem 3.4. *Let π be a conformal anti-invariant $\xi^{\perp}-$submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then any two assertions below imply the third;*

(a) *The distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*

(b) *The map π is a horizontally homothetic submersion.*

(c) $g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi W) = \lambda^2 g_M(-A_X \mathcal{B}Y + \eta(Y)X, \phi W)$

for any $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$.

Proof. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, by the proof of Theorem 3.3, we have

$$g_M(\nabla^M_X Y, W) = g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi W) + \lambda^{-2} g_N(\nabla^{\pi}_X \pi_*\mathcal{C}Y, \pi_*\phi W).$$

Now, if we have (a) and (c), then we obtain

$$-g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(X, \phi W) + g_M(\mathcal{H}\nabla\ln\lambda, \phi W)g_M(X, \mathcal{C}Y) = 0. \tag{3.21}$$

Now, taking $X = \mathcal{C}Y)$ in (3.21) and using (3.17), we get $g_M(\mathcal{H}\nabla\ln\lambda, \phi W)g_M(X, \mathcal{C}Y) = 0$. Hence, λ is a constant on $\Gamma(\phi ker\pi_*)$. On the other hand, taking $X = \phi W$ in (3.21) and using (3.17) we find

$$\begin{aligned} g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(\phi W, \phi W)) &= g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)\{g_M(W, W) - \eta(W)\eta(W)\} \\ &= g_M(\mathcal{H}\nabla\ln\lambda, \mathcal{C}Y)g_M(W, W) = 0 \end{aligned}$$

which means that λ is a constant on $\Gamma(\mu)$. One can easily obtain the other assertions.

From the above theorem, we have the following:

Corollary 3.2. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus <\xi>$. Given $Y, X \in \Gamma((ker\pi_*)^{\perp})$ and $W \in \Gamma(ker\pi_*)$, the following conditions are equivalent to each other;*

(i) *The distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation on the total space.*
(ii) $A_X\mathcal{B}Y = \eta(Y)X$
(iii) $(\nabla\pi_*)(X, \phi W) = -\eta(Y)\pi_*X.$

Now, we investigate the geometry of $ker\pi_*$.

Theorem 3.5. *Let π be a conformal anti-invariant $\xi^{\perp}$−submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then for any $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$ the following conditiones are equivalent to each other;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*
(b) $-\lambda^{-2} g_N(\nabla^N_{\phi_* W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) = g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V\mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W}V)\eta(\mathcal{C}X).$

Proof. Given $V, W \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma((ker\pi_*)^\perp)$, since $g_M(W,\xi) = 0$, by using (2.3) we get $g_M(\nabla^M_V W, \xi) = -g_M(W, \nabla^M_V \xi) = -g_M(W, \phi V) = 0$. Thus we get

$$\begin{aligned} g_M(\nabla^M_V W, X) &= g_M(\phi\nabla^M_V W, \phi X) + \eta(\nabla^M_V W)\eta(X) \\ &= g_M(\phi\nabla^M_V \phi W, \phi X) \\ &= g_M(\nabla^M_V \phi W, \phi X) - g_M((\nabla^M_V \phi)W, \phi X). \end{aligned}$$

Using (2.3), (2.6) and (3.13) we have

$$g_M(\nabla^M_V W, X) = g_M(T_V \phi W, \mathcal{B}X) + g_M(\mathcal{H}\nabla^M_V \phi W, \mathcal{C}X).$$

Since ∇^M is a Levi-Civita connection and $[V, \phi W] \in \Gamma(ker\pi_*)$ we derive

$$g_M(\nabla^M_V W, X) = g_M(T_V \phi W, \mathcal{B}X) + g_M(\nabla^M_{\phi W} V, \mathcal{C}X).$$

Using (2.3), (2.9) and taking into account μ is invariant, we have

$$\begin{aligned} g_M(\nabla^M_V W, X) &= g_M(T_V \phi W, \mathcal{B}X) + g_M(\phi\nabla^M_{\phi W} V, \phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \\ &= g_M(T_V \phi W, \mathcal{B}X) + g_M(\nabla^M_{\phi W} \phi V, \phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X). \end{aligned}$$

Now, using (2.11) and Lemma 2.2 (i) and using the property of π, we obtain

$$\begin{aligned} g_M(\nabla^M_U V, X) = {} & g_M(T_V \phi W, \mathcal{B}X) + \lambda^{-2} g_M(\mathcal{H}\nabla \ln\lambda, \phi W) g_N(\pi_*\phi V, \pi_*\phi\mathcal{C}X) \\ & - \lambda^{-2} g_M(\mathcal{H}\nabla \ln\lambda, \phi V) g_N(\pi_*\phi W, \pi_*\phi\mathcal{C}X) \\ & + g_M(\phi W, \phi V)\lambda^{-2} g_N(\pi_*(\mathcal{H}\nabla \ln\lambda, \pi_*\phi\mathcal{C}X) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_*\phi W} \pi_*\phi V, \pi_*\phi\mathcal{C}X) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \end{aligned}$$

and from Definition 3.1 and (3.17), we have

$$\begin{aligned} g_M(\nabla^M_U V, X) = {} & g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V \mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W} V)\eta(\mathcal{C}X) \\ & + \lambda^{-2} g_N(\nabla^N_{\pi_*\phi W} \pi_*\phi V, \pi_*\phi\mathcal{C}X) \end{aligned}$$

so that we get $(i) \Leftrightarrow (ii)$.

From the above theorem, we deduce:

Theorem 3.6. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^\perp$-submersion. Then, for any $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^\perp)$, any two conditions below imply the third;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*

(b) *The dilation λ is a constant on $\Gamma(\mu)$.*

(c) $-\lambda^{-2}g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X) = g_M(T_V\phi W, \mathcal{B}X) + \eta(\nabla^M_{\phi W}V)\eta(\mathcal{C}X)$.

Proof. Given $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$, by the proof of Theorem (3.5) we have

$$g_M(\nabla^M_W V, X) = g_M(\phi\mathcal{C}X(\ln\lambda)\phi V - T_V\mathcal{B}X, \phi V) + \eta(\nabla^M_{\phi W}V)\eta(\mathcal{C}X)$$
$$+ \lambda^{-2}g_N(\nabla^N_{\pi_*\phi W}\pi_*\phi V, \pi_*\phi\mathcal{C}X).$$

Now, if we have (a) and (c), then we get $g_M(\phi W, \phi V)g_M(\mathcal{H}\nabla\ln\lambda, \phi\mathcal{C}X) = 0$, which means that the dilation λ is a constant on $\Gamma(\mu)$. One can easily obtain the others.

Also we have,

Corollary 3.3. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*)\oplus < \xi >$. Then the following assertiones are equivalent to each other;*

(a) *The distribution $ker\pi_*$ defines a totally geodesic foliation on the total space.*

(b) $T_V\phi W = 0$

for $V, W \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

We note that a differential map π between two Riemannian manifolds is called a totally geodesic map $\Longleftrightarrow$ $(\nabla\pi_*)(Z_1, Z_2) = 0$, for any $Z_1, Z_2 \in \Gamma(TM)$.

Theorem 3.7. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$–submersion. Then π is a totally geodesic map if*

$$-\nabla^N_{\pi_*X}\pi_*Y_2 = \pi_*(\phi(A_X\phi Y_1 + \mathcal{V}\nabla^M_X\mathcal{B}Y_2 + A_X\mathcal{C}Y_2) + \mathcal{C}(\mathcal{H}\nabla^M_X\phi Y_1 + A_X\mathcal{B}Y_2 + \mathcal{H}\nabla^M_X\mathcal{C}Y_2))$$
$$- \eta(Y_2)\pi_*X - \{g_M(X, \phi Y_1) + g_M(X, \mathcal{C}Y_2)\}\pi_*\xi \tag{3.22}$$

for $X \in \Gamma((ker\pi_*)^{\perp}), Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker\pi_*)$ and $Y_2 \in \Gamma((ker\pi_*)^{\perp})$.

Proof. Using (2.2) and (2.11) we have

$$(\nabla\pi_*)(X, Y) = \nabla^N_{\pi_*X}\pi_*Y + \pi_*(-\nabla^M_X Y)$$
$$= \nabla^N_{\pi_*X}\pi_*Y + \pi_*(\phi\nabla_X\phi Y - g(X, \phi Y)\xi - \eta(Y)X)$$

for any $X \in \Gamma((ker\pi_*)^{\perp}), Y \in \Gamma(TM)$. Then by using (2.8), (2.9) and (3.13) we get

$$\begin{aligned}(\nabla\pi_*)(X,Y) &= \nabla^N_{\pi_*X}\pi_*Y_2 + \pi_*(\phi A_X\phi Y_1 + \mathcal{BH}\nabla^M_X\phi Y_1 + \mathcal{CH}\nabla^M_X\phi Y_1 + \mathcal{B}A_X\mathcal{B}Y_2 \\ &\quad + \mathcal{C}A_X\mathcal{B}Y_2 + \phi\mathcal{V}\nabla^M_X\mathcal{B}Y_2 + \phi A_X\mathcal{C}Y_2 + \mathcal{BH}\nabla^M_X\mathcal{C}Y_2 + \mathcal{CH}\nabla^M_X\mathcal{C}Y_2) \\ &\quad - \eta(Y_2)\pi_*X - \{g_M(X,\phi Y_1) + g_M(X,\mathcal{C}Y_2)\}\pi_*\xi\end{aligned}$$

for any $Y = Y_1 + Y_2 \in \Gamma(TM)$, where $Y_1 \in \Gamma(ker\pi_*)$ and $Y_2 \in \Gamma((ker\pi_*)^{\perp})$. Thus taking into account the vertical parts, we obtain

$$\begin{aligned}(\nabla\pi_*)(X,Y) &= \nabla^N_{\pi_*X}\pi_*Y + \pi_*(\phi(A_X\phi Y_1 + \mathcal{V}\nabla^M_X\mathcal{B}Y_2 + A_X\mathcal{C}Y_2) \\ &\quad + \mathcal{C}(\mathcal{H}\nabla^M_X\phi Y_1 + A_X\mathcal{B}Y_2 + \mathcal{H}\nabla^M_X\mathcal{C}Y_2)) \\ &\quad - \eta(Y_2)\pi_*X - \{g_M(X,\phi Y_1) + g_M(X,\mathcal{C}Y_2)\}\pi_*\xi.\end{aligned}$$

Hence $(\nabla\pi_*)(X,Y) = 0 \Longleftrightarrow$ (3.22) is satisfied.

For the totally geodesicness of the map, we also get:

Theorem 3.8. *Let $\pi : (M,\phi,\xi,\eta,g_M) \longrightarrow (N,g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion. π is a totally geodesic map if and only if*

(a) $T_U\phi V = 0$ *and* $\mathcal{H}\nabla^M_U\phi V \in \Gamma(\phi ker\pi_*)$,

(b) *The map π is a horizontally homotetic map,*

(c) $A_Z\phi V = 0$ *and* $\mathcal{H}\nabla^M_Z\phi V \in \Gamma(\phi ker\pi)$

for $X,Y,Z \in \Gamma((ker\pi_*)^{\perp})$ and $U,V \in \Gamma(ker\pi_*)$.

Proof. For any $U,V \in \Gamma(ker\pi_*)$, by using (2.3) and (2.11) we have

$$\begin{aligned}(\nabla\pi_*)(U,V) &= \nabla^N_{\pi_*U}\pi_*V + \pi_*(-\nabla^M_U V) \\ &= \pi_*(\phi\nabla^M_U\phi V - g_M(U,\phi V)\xi - \eta(V)X) \\ &= \pi_*(\phi\nabla^M_U\phi V).\end{aligned}$$

Then from (2.6) and (2.7) we arrive at

$$(\nabla\pi_*)(U,V) = \pi_*(\phi T_U\phi V + \mathcal{CH}\nabla^M_U\phi V).$$

From above equation, $(\nabla\pi_*)(U,V) = 0 \Longleftrightarrow \pi_*(\phi T_U\phi V + \mathcal{CH}\nabla^M_U\phi V) = 0$. Since ϕ is non-singular, $T_U\phi V = 0$ and $\mathcal{H}\nabla^M_U\phi V \in \Gamma(\phi ker\pi_*)$. On the other hand, from Lemma 2.2 we derive

$$(\nabla\pi_*)(X,Y) = X(\ln\lambda)\pi_*Y + Y(\ln\lambda)\pi_*X - g_M(X,Y)\pi_*(\nabla\ln\lambda)$$

for any $X,Y\in\Gamma(\mu)$. It is obvious that if π is a horizontally homotetic map, it follows that $(\nabla\pi_*)(X,Y)=0$. Conversely, if $(\nabla\pi_*)(X,Y)=0$, taking $Y=\phi X$ in the above equation, we get

$$X(\ln\lambda)\pi_*\phi X+\phi X(\ln\lambda)\pi_*X=0.$$

Taking inner product with $\pi_*\phi X$ at the above equation we obtain

$$g_M(\nabla\ln\lambda,X)\lambda^2 g_M(\phi X,\phi X)+g_M(\nabla\ln\lambda,\phi X)\lambda^2 g_M(X,\phi X)=0. \tag{3.23}$$

From (3.23), λ is a constant on $\Gamma(\mu)$. On the other hand, for $U,V\in\Gamma(ker\pi_*)$, from Lemma 2.2 we have

$$(\nabla\pi_*)(\phi U,\phi V)=\phi U(\ln\lambda)\pi_*\phi V+\phi V(\ln\lambda)\pi_*\phi U-g_M(\phi U,\phi V)\pi_*(\nabla\ln\lambda).$$

Again if π is a horizontally homothetic map, then $(\nabla\pi_*)(\phi U,\phi V)=0$. Conversely, if $(\nabla\pi_*)(\phi U,\phi V)=0$, putting U instead of V in above equation, we derive

$$2\phi U(\ln\lambda)\pi_*(\phi U)-g_M(\phi U,\phi U)\pi_*(\nabla\ln\lambda)=0. \tag{3.24}$$

Taking inner product with $\pi_*\phi U$ at (3.24) and since π is a conformal submersion, we have

$$g_M(\phi U,\phi U)\lambda^2 g_M(\nabla\ln\lambda,\phi U)=0$$

which means that the dilation λ is a constant on $\Gamma(\phi ker\pi_*)$. Thus the dilation λ is a constant on $\Gamma((ker\pi_*)^{\perp})$. Now, for $Z\in\Gamma(\mu)$ and $V\in\Gamma(ker\pi_*)$, from (2.3) and (2.11) we have

$$(\nabla\pi_*)(Z,V)=\pi_*(\phi\nabla^M_Z\phi V).$$

In view of (2.8) and (2.9) we have

$$(\nabla\pi_*)(Z,V)=\pi_*(\phi A_Z\phi V+\mathcal{CH}\nabla^M_Z\phi V).$$

Hence $(\nabla\pi_*)(Z,V)=0\Longleftrightarrow\pi_*(\phi A_Z\phi V+\mathcal{CH}\nabla^M_Z\phi V)=0$. Since ϕ is non-singular, $A_Z\phi V=0$ and $\mathcal{H}\nabla^M_Z\phi V\in\Gamma(\phi ker\pi_*)$. Therefore, we obtain the proof.

Now, we give some conditions related to harmonicity of the submersion.

Theorem 3.9. *Let $\pi:(M^{2(m+n)+1},\phi,\xi,\eta,g_M)\longrightarrow(N^{m+2n+1},g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion. Then the tension field τ of π is*

$$\tau(\pi)=-m\pi_*(\mu^{ker\pi_*})+(1-m-2n)\pi_*(\nabla\ln\lambda) \tag{3.25}$$

where $\mu^{ker\pi_*}$ is the mean curvature vector field of the distribution of $ker\pi_*$.

Proof. Let $\{e_1,...,e_m,\phi e_1,...,\phi e_m,\xi,\mu_1,...,\mu_n,\phi\mu_1,...,\phi\mu_n\}$ be orthonormal basis of $\Gamma(TM)$ such that $\{e_1,...,e_m\}$ be orthonormal basis of $\Gamma(ker\pi_*)$, $\{\phi e_1,...,\phi e_m\}$ be orthonormal basis of $\Gamma(\phi ker\pi_*)$ and $\{\xi,\mu_1,...,\mu_n,\phi\mu_1,...,\phi\mu_n\}$ be orthonormal basis of $\Gamma(\mu)$. Then the trace of second fundamental form (restriction to $ker\pi_*\times ker\pi_*$) is given by

$$trace^{ker\pi_*}\nabla\pi_* = \sum_{i=1}^{m}(\nabla\pi_*)(e_i,e_i).$$

Then using (2.11) we obtain

$$trace^{ker\pi_*}\nabla\pi_* = -m\pi_*(\mu^{ker\pi_*}) \tag{3.26}$$

and also, we have

$$trace^{(ker\pi_*)^\perp}\nabla\pi_* = \sum_{i=1}^{m}(\nabla\pi_*)(\phi e_i,\phi e_i) + \sum_{i=1}^{2n}(\nabla\pi_*)(\mu_i,\mu_i) + (\nabla\pi_*)(\xi,\xi).$$

From Lemma 2.2 we get

$$\begin{aligned} trace^{(ker\pi_*)^\perp}\nabla\pi_* &= \sum_{i=1}^{m}2g_M(\mathcal{H}\nabla\ln\lambda,\phi e_i)\pi_*\phi e_i - m\pi_*(\nabla\ln\lambda)\\ &+\sum_{i=1}^{2n}2g_M(\mathcal{H}\nabla\ln\lambda,\mu_i)\pi_*\mu_i - 2n\pi_*(\nabla\ln\lambda)\\ &+2\xi(\ln\lambda)\pi_*\xi - \pi_*(\nabla\ln\lambda). \end{aligned}$$

Since $\{\frac{1}{\lambda(p)}\pi_{*p}(\phi e_i),\frac{1}{\lambda(p)}\pi_{*p}(\mu_h),\frac{1}{\lambda(p)}\pi_{*p}\xi\}_{p\in M,\ 1\leq i\leq m,1\leq h\leq n}$ is an orthonormal basis of $T_{\pi(p)}N$ and using the properties of π, we derive

$$\begin{aligned} trace^{(ker\pi_*)^\perp}\nabla\pi_* &= \sum_{i=1}^{m}2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\phi e_i)\frac{1}{\lambda}\pi_*\phi e_i - m\pi_*(\nabla\ln\lambda)\\ &+\sum_{i=1}^{2n}2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\mu_i)\frac{1}{\lambda}\pi_*\mu_i - 2n\pi_*(\nabla\ln\lambda)\\ &+2g_N(\pi_*\nabla\ln\lambda,\frac{1}{\lambda}\pi_*\xi)\frac{1}{\lambda}\pi_*\xi - \pi_*(\nabla\ln\lambda)\\ &=(1-m-2n)\pi_*(\nabla\ln\lambda) \end{aligned} \tag{3.27}$$

Then proof follows from (3.26) and (3.27).

From the above theorem, we have

Theorem 3.10. *Let $\pi:(M^{2(m+n)+1},\phi,\xi,\eta,g_M)\longrightarrow(N^{m+2n+1},g_N)$ be a conformal anti-invariant $\xi^\perp-$submersion. Then any two conditions below imply the third:*

(a) *The map π is harmonic*

(b) *The fibres are minimal*

(c) *The map π is a horizontally homothetic map.*

Also, we have,

Corollary 3.4. *Let $\pi : (M^{2(m+n)+1}, \phi, \xi, \eta, g_M) \longrightarrow (N^{m+2n+1}, g_N)$ be a conformal anti-invariant $\xi^{\perp}-$submersion. π is harmonic if and only if the fibres are minimal.*

Now, we give some decomposition theorems comes from Theorem 3.3 and Theorem 3.5 in the following:

Theorem 3.11. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion. Then M is a locally product manifold if*

$$-\lambda^{-2} g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi V) = g_M(A_X BY - \mathcal{C}Y(\ln\lambda)X + g_M(X, \mathcal{C}Y)\ln\lambda - \eta(Y)X, \phi V)$$

and

$$-\lambda^{-2} g_N(\nabla^{\pi}_{\phi W} \pi_* \phi V, \pi_* \phi \mathcal{C}X) = g_M(\phi \mathcal{C}X(\ln\lambda)\phi V - T_V \mathcal{B}X, \phi V) + \eta(\nabla_{\phi W} V)\eta(\mathcal{C}X)$$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U, V \in \Gamma(ker\pi_*)$, where $M_{(\ker\pi_*)^{\perp}}$ and $M_{(\ker\pi_*)}$ are integral manifolds of the distributions $(\ker\pi_*)^{\perp}$ and $(\ker\pi_*)$. Conversely, if M is a locally product manifold of the form $M_{(\ker\pi_*)^{\perp}} \times M_{(\ker\pi_*)}$ then we have

$$\lambda^{-2} g_N(\nabla^{\pi}_X \pi_* \mathcal{C}Y, \pi_* \phi V) = g_M(\mathcal{C}Y(\ln\lambda)X - g_M(X, \mathcal{C}Y)\ln\lambda + \eta(Y)X, \phi V)$$

and

$$-\lambda^{-2} g_N(\nabla^{\pi}_{\phi W} \pi_* \phi V, \pi_* \phi \mathcal{C}X) = g_M(\phi \mathcal{C}X(\ln\lambda)\phi V, \phi V) + \eta(\nabla_{\phi W} V)\eta(\mathcal{C}X).$$

From Corollary 3.2 and Corollary 3.3, we have

Theorem 3.12. *Let $\pi : (M, \phi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a conformal anti-invariant $\xi^{\perp}$-submersion with $(ker\pi_*)^{\perp} = \phi(ker\pi_*) \oplus < \xi >$. Then M is a locally product manifold if $A_X \mathcal{B}Y = \eta(Y)X$ and $T_V \phi W = 0$ for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$.*

Acknowledgement

The authors are grateful to the referee for his/her valuable comments and suggestions.

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International Journal of Maps in Mathematics
Volume:1, Issue:1, (2018), Pages:68-79
www.journalmim.com

ALMOST POLY-NORDEN MANIFOLDS

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Abstract. In this paper, poly-Norden manifolds are introduced as a new type of manifolds. We show that this class includes Norden manifolds and Euclidean $n-$ space as examples. We investigate certain geometric properties of poly-Norden manifolds and obtain conditions for a holomorphic-like map between such manifolds to be totally geodesic. We also investigate constancy of certain maps between poly-Norden manifolds and other manifolds endowed with differential structures.

1. Introduction

Manifolds equipped with certain differential-geometric structures have been studied widely in differential geometry. Indeed, almost complex manifolds, almost contact manifolds and almost product manifolds have been studied extensively by many authors. Recently, by inspiring the Golden ratio, Golden Riemannian manifolds were introduced in [5]. In [11], the authors have also introduced metallical Riemannian manifolds by inspiring metallic mean which is a generalization of Golden mean, Silver mean etc. Both manifolds have been studied by many authors [6], [7],[9],[10], [11], [12],[15], [14], [16] and [19]. We note that metallic mean was first defined by de Spinadel [17].

Received: 2018-02-14 *Revised:2018-03-13* *Accepted:2018-03-21*

2010 Mathematics Subject Classification: 53C40,53C15, 55N25, 53C65.

Key words: *Golden Mean, Metallic Mean, Silver Mean, Bronze Mean, Lucas number, poly-Norden structure, poly-Norden manifold.*

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On the other hand, in [13], the author has defined Bronze mean which is different from the Bronze mean given in [11]. We also note that there is no inclusion relation between the Bronze mean defined in [13] and metallic mean. In [13], the author introduced the Bronze Fibonacci and Lucas numbers. He also investigated the relationship between the convergents of the continued fractions of the powers of the Bronze means and the Bronze Fibonacci and Lucas numbers.

In this paper, by inspiring from [5] and [13], we introduce almost poly-Norden manifolds and investigate the geometry of such manifolds. First of all, we observe that such manifolds include some well known manifolds; Norden manifolds and Euclidean spaces. We then investigate certain properties of this structure and obtain constancy of certain maps.

2. Preliminaries

In this section, we gather main tools needed for the paper.

2.1. **A new Bronze mean.** We will not repeat Golden ratio, Fibonacci number or Lucas numbers which are well known. But we will give brief information on a new mean and related numbers introduced in [13]. The Bronze mean is defined by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2} \tag{2.1}$$

which is the positive solution of the equation

$$x^2 - mx + 1 = 0.$$

The Bronze Fibonacci numbers $(f_{m,n})$ are a family of sequences defined by the recurrence $f_{m,n+2} = mf_{m,n+1} - f_{m,n}$, where $f_{m,0} = 0$ and $f_{m,1} = 1$. The Bronze Lucas numbers $(l_{m,n})$ are a family of sequence defined by the recurrence $l_{m,n+2} = ml_{m,n+1} - l_{m,n}$, where $l_{m,0} = 2$ and $l_{m,1} = m$. The continued fractions for the Bronze means are $\{m-1; \overline{1, m-2}\}$. The recurrence relation $B_m^{n+2} = mB_m^{n+1} - B_m^n$ is satisfied. The relations between Bronze Fibonacci numbers and Bronze Lucas numbers are

$$B_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2 - 4}}{2}.$$

Also note that the convergents of B_m^a are $\frac{f_{m,a(n+1)}}{f_{m,an}}$. Proofs of all above statements and more results can be found in [13].

2.2. **Totally geodesic maps.** Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth mapping between them. Then the differential $d\varphi$ of φ can be viewed a section of the bundle $Hom\left(TM, \varphi^{-1}TN\right) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibres $\left(\varphi^{-1}TN\right)_p = T_{\varphi(p)}N$, $p \in M$. $Hom\left(TM, \varphi^{-1}TN\right)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$\nabla d\varphi\,(X, Y) = \nabla^{\varphi}_X d\varphi\,(Y) - d\varphi\left(\nabla^M_X Y\right) \tag{2.2}$$

for $X, Y \in \Gamma(TM)$, where $\Gamma(TM)$ denotes the Lie algebra of the vector fields on M . It is known that the second fundamental form is symmetric. A smooth map $\varphi : (M, g_M) \to (N, g_N)$ is said to be totally geodesic if $\nabla d\varphi = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. For more information, see [1].

3. Almost poly-Norden manifolds

By inspiring from the Bronze mean (2.1) we introduce a new structure on a differentiable manifold M, namely, poly-Norden structure.

Definition 3.1. *Let M be a differentiable manifold. A poly-Norden structure on M is an $(1,1)$ tensor field Φ which satisfies the equation*

$$\Phi^2 = m\Phi - I \tag{3.3}$$

where I is the identity operator on the Lie algebra $\chi(M)$ of the vector fields on M. In this case, (M, Φ) is called an almost poly-Norden manifold.

We give an example of almost poly-Norden manifolds.

Example 3.1. *Consider the 4−tuples real space $\mathbb{R}^4$ and define a map by*

$$\begin{array}{cccc} \Phi : & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ & (x_1, x_2, y_1, y_2) & & (B_m x_1, B_m x_2, \bar{B}_m y_1, \bar{B}_m y_2), \end{array}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$ and $\bar{B}_m = m - B_m$. Then it is easy to see that Φ satisfies $\Phi^2 = m\Phi - I$. Thus $(\mathbb{R}^4, \Phi)$ is an example of almost poly-Norden manifold.

We say that a semi-Riemannian metric g is $\Phi-$ compatible if

$$g(\Phi X, \Phi Y) = mg(\Phi X, Y) - g(X, Y) \tag{3.4}$$

for every $X, Y \in \chi(M)$. From this it follows that Φ is a self-adjoint operator with respect to g, i.e.,

$$g(\Phi X, Y) = g(X, \Phi Y) \tag{3.5}$$

Definition 3.2. *A semi-Riemannian manifold (M, g) endowed with a poly-Norden structure Φ so that the semi-Riemannian metric g is Φ-compatible is named an almost poly-Norden semi-Riemannian manifold and (g, Φ) is called an almost poly-Norden Riemannian structure on M.*

We now give an example of almost poly-Norden semi-Riemannian manifolds.

Example 3.2. *Let M be an almost complex manifold with almost complex structure J. A metric g is a Norden metric if $g(JX, JY) = -g(X, Y)$. If (M^{2n}, J) is an almost complex manifold with Norden metric g, Then (M^{2n}, J, g) is called an almost Norden manifold. Thus every almost Norden manifold is an almost poly-Norden semi-Riemannian manifold with $m = 0$.*

From now on, we will assume that the number m is different from zero throughout the article. We now investigate the geometry of an almost poly-Norden manifold.

Proposition 3.1. *Eigenvalues of an almost poly-Norden structure Φ are $\frac{m+\sqrt{m^2-4}}{2}$ and $\frac{m-\sqrt{m^2-4}}{2}$.*

Proposition 3.2. *An almost poly-Norden structure Φ is an isomorphism on a tangent space of M*

Since Φ is isomorphism, it has an inverse. Let us denote the inverse of Φ by $\tilde{\Phi}$, then we have

$$\tilde{\Phi} = -\Phi + mI. \tag{3.6}$$

We also have the following result.

Proposition 3.3. *$\tilde{\Phi}$ is not an poly-Norden structure on M.*

The following result shows that an almost complex structure determines a poly-Norden structure and vice versa.

Proposition 3.4. *Every complex structure on a semi-Riemannian manifold induces two poly-Norden structures given by*

$$\Phi_1 = \frac{m}{2}I + \frac{\sqrt{4-m^2}}{2}J,\ \Phi_2 = \frac{m}{2}I - \frac{\sqrt{4-m^2}}{2}J,\ -2 < m < 2$$

Proposition 3.5. *Every poly-Norden structure on a semi-Riemannian manifold induces two almost complex structures given by*

$$J_1 = \frac{-m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\Phi,\ J_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\Phi,\ -2 < m < 2$$

Next result implies that there are two orthogonal complementary distributions on almost polygonal semi-Riemannian manifold (M, ϕ, g).

Proposition 3.6. *On an almost poly-Norden semi-Riemannian manifold (M, ϕ, g), there are two complementary distributions D_l and $D_{l^\perp}$ corresponding to the projection operators*

$$l = \frac{1}{\sqrt{m^2-4}}(B_m I - \Phi),\ l^\perp = \frac{1}{\sqrt{m^2-4}}(-\bar{B}_m I + \Phi) \tag{3.7}$$

Corollary 3.1. *The complementary distributions D_l and $D_{l^\perp}$ are orthogonal with respect to the $\Phi-$compatible metric g, i.e. $g(D_l, D_{l^\perp}) = 0$.*

Definition 3.3. *Let (M, Φ, g) be an almost poly-Norden semi-Riemannian manifold. If the almost poly-Norden structure is parallel with respect to the Levi-Civita connectin ∇. Then (M, Φ, g) is called a poly-Norden semi-Riemannian manifold.*

One can see that $\nabla\Phi =$ is equivalent to $N_\Phi = 0$, where N_Φ is the Nijenhuis tensor field with respect to Φ, see:[3], [4], [8].

4. Poly-Norden maps between poly-Norden manifolds

In this section we give a new notion, namely poly-Norden map, and investigate conditions for a poly-Norden map to be totally geodesic. From now on we frequently denote an almost poly-Norden manifold by (M, Φ, m). We first give the following definion which is a version of a holomorphic map. Let φ be a map from an almost poly-Norden manifold (M, Φ_M, m) to an almost poly-Norden manifold (N, Φ_N, m'). Then we say that φ is a poly-Norden map if it satisfies $d\varphi\Phi_M = \Phi_N d\varphi$, where $d\varphi$ denotes the derivative map of φ. We provide the following elementary example.

Example 4.1. *Let $\varphi : \mathbb{R}^4 \to \mathbb{R}^2$ be a map defined by $\varphi(x_1, x_2, x_3, x_4) = \left(\frac{x_1+x_2}{4}, \frac{x_3+x_4}{4}\right)$. Then, by direct calculations*

$$\ker d\varphi = span\left\{X_1 = \partial_{x_1} - \partial_{x_2}, X_2 = \partial_{x_3} - \partial_{x_4}\right\}$$

and

$$(\ker d\varphi)^{\perp} = span\left\{Z_1 = \partial_{x_1} + \partial_{x_2}, Z_2 = \partial_{x_3} + \partial_{x_4}\right\}.$$

Then considering poly-Norden strucutures on $\mathbb{R}^4$ and $\mathbb{R}^2$ defined by

$$\Phi(x_1, x_2, x_3, x_4) = \left(B_m x_1, B_m x_2, \bar{B}_m x_3, \bar{B}_m x_4\right)$$

and

$$\Phi'(a_1, a_2) = \left(B_m a_1, \bar{B}_m a_2\right)$$

where B_m and $\bar{B}_m$ are eigenvalues of poly-Norden structures. It is easy to see that $d\varphi\,(\Phi Z_1) = \Phi' d\varphi\,(Z_1)$ and $d\varphi\,(\Phi Z_2) = \Phi' d\varphi\,(Z_2)$. Thus φ is a poly-Norden map.

From now on, when we mention a poly-Norden semi-Riemannian manifold in this section, we will assume that its almost poly-Norden structure is integrable.

Lemma 4.1. *Let φ be a poly-Norden map from a poly-Norden semi-Riemannian manifold (M, Φ, g_M, m) to a poly-Norden semi-Riemannian manifold (N, Φ', g', m') such that $d\varphi\,\Phi = \Phi' d\varphi$ is satisfied. Then we have*

$$(\nabla d\varphi)(\Phi X, \Phi Y) = m'\Phi'(\nabla d\varphi)(X, Y) - (\nabla d\varphi)(X, Y) \tag{4.8}$$

for $X, Y \in \Gamma(TM)$.

Proof. For $X, Y \in \Gamma(TM)$, from (2.13) and (2.11) we have

$$(\nabla d\varphi)(X, \Phi Y) = \nabla_X^{\varphi} d\varphi\,(\Phi Y) - d\varphi\left(\nabla_X^M \Phi Y\right).$$

Since $d\varphi\Phi = \Phi' d\varphi$ and both Φ and Φ' are integrable we have

$$(\nabla d\varphi)(X, \Phi Y) = \Phi'(\nabla d\varphi)(X, Y).$$

Using this equation and (3.3) we have the assertion.

We now give a necessary and sufficient condition for φ to be totally geodesic. We recall that a map φ is totally geodesic if $\nabla d\varphi = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths. From Lemma 4.1, we have the following result.

Theorem 4.1. *Let φ be an poly-Norden map from a poly-Norden semi-Riemannian manifold (M, Φ, g) to a poly-Norden semi-Riemannian manifold (N, Φ', g'). If one of the following conditions is satisfied, then φ is totally geodesic;*

(1) $(\nabla d\varphi)(\Phi X, \Phi Y) = m'\Phi'(\nabla d\varphi)(X, Y)$, $\forall X, Y \in \Gamma(TM)$,
(2) $(\nabla d\varphi)(\Phi X, \Phi Y) = (\nabla d\varphi)(X, Y)$, $\forall X, Y \in \Gamma(TM)$ *and* $m' \neq 0$.

5. Certain maps between almost poly-Norden manifolds and manifolds endowed with differentiable structures

For maps between differentiable manifolds, authors normally study such maps under certain conditions imposed on the manifolds and maps. A crucial question is that whether there exist such maps under the restrictions. Therefore, in this section, we investigate the existence of holomorphic-like maps from (into) poly-Norden manifolds to manifolds endowed with differentiable structure such as almost Golden structure, almost complex structure, almost product structure and almost contact structure. We show that such maps defined between almost poly-Norden manifolds and manifolds endowed with differentiable structures are constant under some assumptions.

5.1. **Maps between almost poly-Norden manifolds and almost Golden manifolds.** Let $\bar{M}$ be a differentiable manifold. A golden structure on $\bar{M}$ is an $(1, 1)$ tensor field P which satisfies the equation

$$P^2 = P + I \tag{5.9}$$

where I is the identity transformation. In this case P is called an almost Golden structure and (M, P) is called almost Golden manifold. We say that the metric g is P compatible if

$$g\left(PX, Y\right) = g\left(X, PY\right) \tag{5.10}$$

for all $X, Y \in \Gamma\left(T\bar{M}\right)$. If we substitute PX into X in (2.12) the equation (2.12) may also written as

$$g\left(PX, PY\right) = g\left(P^2X, Y\right) = g\left(\left(P + I\right)X, Y\right) = g\left(PX, Y\right) + g\left(X, Y\right)$$

The Riemannian metric (2.12) is called $P-$compatible and $\left(\bar{M}, P, g\right)$ is named a Golden Riemannian manifold [5]. It is known[5] that a Golden structure φ is integrable if the Nijenhuis tensor N_φ vanishes. In [10], the authors show that a Golden structure is integrable if and only if $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.

Theorem 5.1. *Let φ be a smooth map from an almost Golden manifold $\left(\bar{M}, P,\right)$ to an almost poly-Norden manifold (M', Φ, m) such that the condition $d\varphi P = \Phi d\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp\sqrt{5}$.*

Proof. Let $\left(\bar{M}, P\right)$ be an almost Golden manifold and (M', Φ, m) a almost poly-Norden manifold. Then apply Φ to equation $d\varphi P = \Phi d\varphi$ and using (3.1) and (2.11), we get

$$(1-m)\Phi d\varphi(X) = -2d\varphi\left(X\right) \tag{5.11}$$

for $X \in \Gamma\left(T\bar{M}\right)$. Applying Φ to (5.11) again, we derive

$$(-m^2+m+2)\Phi d\varphi\left(X\right) = (1-m)d\varphi\left(X\right) \tag{5.12}$$

Then (5.11) and (5.12) imply that

$$(5-m^2)\Phi d\varphi(X) = 0$$

which gives our assertion.

In a similar way, we have the following result.

Theorem 5.2. *Let φ be a smooth map from an almost poly-Norden manifold (M', Φ, m) to an almost Golden manifold $\left(\bar{M}, P\right)$ such that the condition $d\varphi\Phi = Pd\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp\sqrt{5}$.*

5.2. **Maps between almost poly-Norden manifolds and almost complex manifolds.** Let M' be a $2n-$ dimensional real manifold. An almost complex structure J on M' is a $(1,1)$ tensor field such that

$$J^2 = -I \tag{5.13}$$

where I is the identity transformation. Then (M', J) is called almost complex manifold [18].

Theorem 5.3. *Let φ be a smooth map from an almost poly-Norden manifold $\left(\bar{M}, \Phi, m\right)$ to an almost complex manifold(M', J) such that the condition $d\varphi\Phi = Jd\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, \Phi, m\right)$ be an almost poly-Norden manifold and (M', J) an almost complex manifold. Suppose that $\varphi : \bar{M} \to M'$ satisfies $d\varphi\left(\Phi X\right) = Jd\varphi\left(X\right), \quad X \in \Gamma\left(T\bar{M}\right)$. Then apply J to above equation and using (2.1) and (2.11), we get

$$d\varphi\left(m\Phi X\right) - d\varphi\left(X\right) = -d\varphi\left(X\right), \quad X \in \Gamma\left(T\bar{M}\right). \tag{5.14}$$

From (5.14) we obtain $mJd\varphi(X) = 0$ which shows that φ is constant due to J is nonsingular.

In a similar way, we have the following result.

Theorem 5.4. *Let φ be a smooth map from an almost complex manifold (M', J) to an almost poly-Norden manifold $\left(\bar{M}, \Phi, m\right)$ such that the condition $d\varphi J = \Phi d\varphi$ is satisfied. Then φ is a constant map.*

5.3. **Maps between almost poly-Norden manifolds and almost product manifolds.** Let N be an $n-$dimensional manifold with a tensor of type (1.1) such that

$$F^2 = I, \tag{5.15}$$

where I is the identity transformation. Then we say that N is an almost product manifold with almost product structure F. We put

$$Q = \frac{1}{2}(I + F), \quad Q^{'} = \frac{1}{2}(I - F). \tag{5.16}$$

Then we have

$$Q + Q^{'} = I, \;\; Q^2 = Q, \;\; Q^{'2} = Q^{'}, \;\; QQ^{'} = Q'Q = 0 \tag{5.17}$$

and

$$F = Q - Q'. \tag{5.18}$$

for details, see:[18].

For a holomorphic-like map between poly-norden manifolds and almost product manifolds, we have the following result.

Theorem 5.5. *Let φ be a smooth map from an almost poly-Norden manifold (M', Φ, m) to an almost product manifold $\left(\bar{M}, F\right)$ such that the condition $d\varphi\Phi = F d\varphi$ is satisfied. Then φ is a constant map if $m \neq \mp 2$.*

Proof. Applying F to $d\varphi\Phi = F d\varphi$ and using (3.3) and (5.15) we have

$$md\varphi(\Phi X) = 2d\varphi(X) \tag{5.19}$$

for $X \in \Gamma(TM)$. Applying F to (5.19) and using (3.3) we obtain

$$(m^2 - 2)d\varphi(\Phi X) = md\varphi(X). \tag{5.20}$$

Thus from (5.19) and (5.20) we get

$$(m^2 - 4)d\varphi(X) = 0$$

which gives proof.

In a similar way, we have the following result.

Theorem 5.6. *Let φ be a smooth map from an almost product manifold $\left(\bar{M},F\right)$ to an almost poly-Norden manifold (M',Φ,m) such that the condition $d\varphi F=\Phi d\varphi$ is satisfied. Then φ is a constant map if $m\neq\mp 2$.*

5.4. **Maps between almost poly-Norden manifolds and almost contact manifolds.** An $n-$dimensional differentiable manifold $\bar{M}$ is said to have an almost contact structure $(\bar{\varphi},\xi,\eta)$ if it carries a tensor field $\bar{\varphi}$ of type $(1,1)$, a vector field ξ and $1-$ form η on $\bar{M}$ respectively such that

$$\bar{\varphi}^2=-I+\eta\otimes\xi,\bar{\varphi}\xi=0,\eta\circ\bar{\varphi}=0,\eta\left(\xi\right)=1 \tag{5.21}$$

where I is the identity transformation [2].

Theorem 5.7. *Let φ be a smooth map from an almost contact manifold $\left(\bar{M},\bar{\varphi},\eta,\xi\right)$ to an almost poly-Norden manifold (M',Φ,m) such that the condition $d\varphi\bar{\varphi}=\Phi d\varphi$ is satisfied. Then φ is a constant map.*

Proof. Using (3.3) and (5.21) we have

$$m\Phi d\varphi(X)=\eta(X)d\varphi(\xi) \tag{5.22}$$

for $X\in\Gamma(T\bar{M})$. Using the second relation of (5.21) in (5.22) and applying (3.3) we find

$$m\Phi d\varphi(X)=d\varphi(X). \tag{5.23}$$

Using once again (3.3) and the relation $d\varphi\bar{\varphi}=\Phi d\varphi$ we get

$$(m^2-1)\Phi d\varphi(X)=md\varphi(X). \tag{5.24}$$

Thus from (5.23) and (5.24) we obtain $d\varphi(X)=0$, $\forall X\in\Gamma(T\bar{M})$.

In a similar way, we have the following result.

Theorem 5.8. *Let φ be a smooth map from an almost poly-Norden manifold (M',Φ,m) to an almost contact manifold $\left(\bar{M},\bar{\varphi},\eta,\xi\right)$ such that the condition $d\varphi\Phi=\bar{\varphi}d\varphi$ is satisfied. Then φ is a constant map.*

Remark 5.1. *In this paper, we introduce a new manifold defined by a new mean given in* [13]*. As we have seen, this new manifold has rich geometric properties and it is also useful to characterize certain maps. Therefore, we invite readers to explore further geometric properties of this new class.*

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International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:80-90

www.journalmim.com

OPERATORS ASSOCIATED WITH OF GOLDEN RIEMANNIAN STRUCTURES ON TANGENT AND COTANGENT BUNDLES

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ABSTRACT. In this paper, operators were applied to vertical and horizontal lifts with respect to the Golden Riemannian structures on tangent and cotangent bundles, respectively.

1. INTRODUCTION

For a manifold M, let φ be a $(1,1)-$tensor field on M. If the polynomial $X^2 - X - 1$ is the minimal polynomial for a structure φ satisfying $\varphi^2 - \varphi - 1 = 0$, then φ is called a Golden structure on M and (M, φ) is a Golden manifold [1, 5, 6]. This structure was inspired by the Golden Ratio, which was described by Johannes Kepler $(1571 - 1630)$. The number $\eta = \frac{1+\sqrt{5}}{2} \approx 1.618...$, which is a solution of the equation $x^2 - x - 1 = 0$, is the Golden ratio. We note that for Golden structures, $\varphi \neq aI$, where $a \in R$. If $\varphi = aI$, $a = \frac{1+\sqrt{5}}{2}$, then its minimal polynomial is $X - a$. However, the minimal polynomial of the Golden structure φ is $X^2 - X - 1$.

Let (M, g) be a Riemannian manifold endowed with the Golden structure φ such that [1, 5, 6]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1.1}$$

Received:2018-03-12 *Accepted:2018-03-30*

2010 Mathematics Subject Classification: 15A72, 47B47, 53A45, 53C15.

Key words: Tachibana operators, Vishnevskii operators, Horizontal lift, Vertical lift, Golden Riemannian structures, Tangent bundle, Cotangent bundle.

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for all $X,Y\in\Im_0^1(M)$. If we substitute φX into X in (1.1), the equation (1.1) may also be written as

$$g(\varphi X,\varphi Y)=g\left(\varphi^2X,Y\right)=g((\varphi+1)X,Y)=g(\varphi X,Y)+g(X,Y). \tag{1.2}$$

The Riemannian metric (1.1) is called φ-compatible and (M,φ,g) is named Golden Riemannian manifold. Such Riemannian metrics are also referred to as pure metrics [7, 11].

Let φ be a (1.1)-tensor field on M, i.e. $\varphi\in\Im_1^1(M)$. A tensor field t of type (r,s) is called a pure tensor field with respect to φ if

$$\begin{aligned} t\left(\varphi X_1,X_2,...,X_s;\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}\right) &= t\left(X_1,\varphi X_2,...,X_s;\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}\right)\\ &\quad ...\\ &= t\left(X_1,X_2,...,\varphi X_s;\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}\right)\\ &= t\left(X_1,X_2,...,X_s;{}'\varphi\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}\right)\\ &= t\left(X_1,X_2,...,X_s;\overset{1}{\xi},\overset{2}{\xi},...,{}'\varphi\overset{r}{\xi}\right)\end{aligned}$$

for any $X_1,X_2,...,X_s\in\Im_0^1(M)$ and $\overset{1}{\xi},\overset{2}{\xi},...,\overset{r}{\xi}\in\Im_1^0(M)$, where ${}'\varphi$ is the adjoint operator of φ defined by

$$\left({}'\varphi\xi\right)(X)=\xi(\varphi X)=(\xi\circ\varphi)(X),X\in\Im_0^1(M),\xi\in\Im_1^0(M).$$

We define an operator

$$\phi_\varphi:\Im_s^0(M)\to\Im_{s+1}^0(M)$$

applied to the pure tensor field t of type $(0,s)$ with respect to φ by [12]

$$\begin{aligned}(\phi_\varphi t)(X,Y_1,...,Y_s) &= (\varphi X)t(Y_1,...,Y_s)-Xt(\varphi Y_1,...,Y_s)\\ &\quad+\sum_{\lambda=1}^{s}t(Y_1,...,(L_{Y_\lambda}\varphi)X,...,Y_s)\end{aligned}$$

for any $X,Y_1,...,Y_s\in\Im_0^1(M)$, where L_Y denotes the Lie differentiation with respect to Y.

1.1. **The Golden structure on tangent bundle $T(M)$.** Golden structure on a Riemannian manifold is important because this structure has relation with pure Riemannian metrics with respect to the structure. Pure metrics with respect to certain structures were studied by various authors (for example see [7, 8], etc.). Since Riemannian Golden and almost product structures are related to each other (see Theorem 2.3 in [4]), the method of $\phi-$operator used in the theory of almost product structures can be transferred to Golden structures.

Definition 1.1. *The Sasaki metric on the tangent bundle $T(M)$ is defined by*

$$^{S}g\left(X^{H},Y^{H}\right)=\left(g\left(X,Y\right)\right)^{V}, \tag{1.3}$$

$$^{S}g\left(X^{V},Y^{H}\right)={}^{S}g\left(X^{H},Y^{V}\right)=0 \tag{1.4}$$

$$^{S}g\left(X^{V},Y^{V}\right)=\left(g\left(X,Y\right)\right)^{V}, \tag{1.5}$$

for all $X,Y\in\Im_{0}^{1}(M)$ (see [13], p. 155-175). It is obvious that the Sasaki metric ^{S}g is contained in the class of the so-called $g-$natural metrics on the tangent bundle.

Definition 1.2. *The Golden structure $\widetilde{J}$ on tangent bundle $T(M)$, which implies $\widetilde{J}^{2}-\widetilde{J}-I=0$, defined by* [4]

$$\begin{array}{rcl}\widetilde{J}\left(X^{H}\right)&=&\frac{1}{2}\left(X^{H}+\sqrt{5}X^{V}\right)\\ \widetilde{J}\left(X^{V}\right)&=&\frac{1}{2}\left(X^{V}+\sqrt{5}X^{H}\right)\end{array} \tag{1.6}$$

for all $X,Y\in\Im_{0}^{1}(M)$.

Theorem 1.1. *Let (M,g) be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the Sasaki metric ^{S}g and the Golden structure $\widetilde{J}$ defined by (1.6). The triple $\left(T(M),\widetilde{J},{}^{S}g\right)$ is a Golden Riemannian manifold.*

1.2. **The Golden structure on cotangent bundle $T^{*}(M)$.**

Definition 1.3. *A Sasakian metric ^{S}g is defined on $T^{*}(M)$ by the three equations*

$$^{S}g\left(\omega^{V},\theta^{V}\right)=\left(g^{-1}\left(\omega,\theta\right)\right)^{V}=g^{-1}\left(\omega,\theta\right)\circ\pi, \tag{1.7}$$

$$^{S}g\left(\omega^{V},Y^{H}\right)=0, \tag{1.8}$$

$$^{S}g\left(X^{H},Y^{H}\right)=\left(g\left(X,Y\right)\right)^{V}=g\left(X,Y\right)\circ\pi \tag{1.9}$$

for any $X,Y\in\Im_{0}^{1}(M)$ and $\omega,\theta\in\Im_{1}^{0}(M)$. Since any tensor field of type $(0,2)$ on $T^{*}(M)$ is completely determined by its action on vector fields of type X^{H} and ω^{V} (see [13], p. 280), it follows that ^{S}g is completely determined by the equations (1.7),(1.8) and (1.9).

Definition 1.4. *The Golden structure $\tilde{\varphi}$ on $T^{*}(M)$ defined by* [4]

$$\begin{aligned} \tilde{\varphi}X^{H} &= \frac{1}{2}\left(X^{H}+\sqrt{5}\tilde{X}^{V}\right) \\ \tilde{\varphi}\omega^{V} &= \frac{1}{2}\left(\omega^{V}+\sqrt{5}\tilde{\omega}^{H}\right) \end{aligned} \tag{1.10}$$

for any $X\in\Im_{0}^{1}(M)$ and $\omega\in\Im_{1}^{0}(M)$, where $\tilde{X}=g\circ X\in\Im_{1}^{0}(M)$, $\tilde{\omega}=g^{-1}\circ\omega\in\Im_{0}^{1}(M)$. Also note that ${}^{S}g$ is pure with respect to $\tilde{\varphi}$.

2. Main Results

2.1. The Tachibana operators applied to vertical and horizontal lifts with respect to the Golden structure on tangent bundle and cotangent bundle.

Definition 2.1. *Let $\varphi\in\Im_{1}^{1}(M)$, and $\Im(M)=\sum_{r,s=0}^{\infty}\Im_{s}^{r}(M)$ be a tensor algebra over R. A map $\phi_{\varphi}\mid_{r+s\rangle 0}:\overset{*}{\Im}(M)\to\Im(M)$ is called a Tachibana operator or ϕ_{φ} operator on M if*

$a)$ ϕ_{φ} is linear with respect to constant coefficient,

$b)$ $\phi_{\varphi}:\overset{*}{\Im}(M)\to\Im_{s+1}^{r}(M)$ for all r and s,

$c)$ $\phi_{\varphi}(K\overset{C}{\otimes}L)=(\phi_{\varphi}K)\otimes L+K\otimes\phi_{\varphi}L$ for all $K,L\in\overset{*}{\Im}(M)$,

$d)$ $\phi_{\varphi X}Y=-(L_{Y}\varphi)X$ for all $X,Y\in\Im_{0}^{1}(M)$ where L_{Y} is the Lie derivation with respect to Y,

$e)$

$$\begin{aligned} (\phi_{\varphi X}\eta)Y &= (d(\imath_{Y}\eta))(\varphi X)-(d(\imath_{Y}(\eta o\varphi)))X+\eta((L_{Y}\varphi)X) \\ &= \phi X(\imath_{Y}\eta)-X(\imath_{\varphi Y}\eta)+\eta((L_{Y}\varphi)X) \end{aligned} \tag{2.11}$$

for all $\eta\in\Im_{1}^{0}(M)$ and $X,Y\in\Im_{0}^{1}(M)$, where $\imath_{Y}\eta=\eta(Y)=\eta\overset{C}{\otimes}Y,\overset{*}{\Im}{}_{s}^{r}(M)$ the module of all pure tensor fields of type (r,s) on M according to the affinor field φ [2, 3, 9](see [10] for applied to pure tensor field).

Theorem 2.1. *Let L_{X} the operator Lie derivation with respect to X , $\widetilde{J}$ be the Golden structure on tangent bundle $T(M)$, which implies $\widetilde{J}^{2}-\widetilde{J}-I=0$, defined by (1.6) and $\phi_{\widetilde{J}}$*

the Tachibana operator on M. We get the following formulas

$$\begin{aligned} i)\ \phi_{\tilde{J}X^V}Y^H &= \frac{\sqrt{5}}{2}((R(Y,X)U)^V+(\nabla_X Y)^H), \\ ii)\ \phi_{\tilde{J}X^H}Y^H &= -\frac{\sqrt{5}}{2}((\nabla_X Y)^V+(R(Y,X)U)^H), \\ iii)\ \phi_{\tilde{J}X^V}Y^V &= \frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^V, \\ iv)\ \phi_{\tilde{J}X^H}Y^V &= -\frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^H, \end{aligned}$$

where R is the curvature tensor of ∇, $X,Y\in\Im_0^1(M)$ and $\widetilde{J}\in\Im_1^1(M)$.

Proof. $i)$

$$\begin{aligned} \phi_{\tilde{J}X^V}Y^H &= -\left(L_{Y^H}\tilde{J}\right)X^V=-L_{Y^H}\tilde{J}X^V+\tilde{J}L_{Y^H}X^V \\ &= -L_{Y^H}\frac{1}{2}\left(X^V+\sqrt{5}X^H\right)+\tilde{J}\left(\tilde{\nabla}_Y X\right)^V \\ &= -\frac{1}{2}\left[Y^H,X^V\right]-\frac{\sqrt{5}}{2}\left[Y^H,X^H\right]+\frac{1}{2}\left(\tilde{\nabla}_Y X\right)^V+\frac{\sqrt{5}}{2}\left(\tilde{\nabla}_Y X\right)^H \\ &= -\frac{1}{2}\left(\tilde{\nabla}_Y X\right)^V-\frac{\sqrt{5}}{2}\left([Y,X]^H-(R(Y,X)U)^V\right)+\frac{1}{2}\left(\tilde{\nabla}_Y X\right)^V \\ &\quad +\frac{\sqrt{5}}{2}\left(\tilde{\nabla}_Y X\right)^H \\ &= -\frac{\sqrt{5}}{2}[Y,X]^H+\frac{\sqrt{5}}{2}(R(Y,X)U)^V+\frac{\sqrt{5}}{2}((\nabla_X Y)^H+[Y,X]^H) \\ &= \frac{\sqrt{5}}{2}((R(Y,X)U)^V+(\nabla_X Y)^H) \end{aligned}$$

$ii)$

$$\begin{aligned} \phi_{\tilde{J}X^H}Y^H &= -\left(L_{Y^H}\tilde{J}\right)X^H=-L_{Y^H}\tilde{J}X^H+\tilde{J}L_{Y^H}X^H \\ &= -L_{Y^H}(\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V)+\tilde{J}([Y,X]^H-(R(Y,X)U)^V) \\ &= -\frac{1}{2}\left[Y^H,X^H\right]-\frac{\sqrt{5}}{2}\left[Y^H,X^V\right]+\tilde{J}[Y,X]^H-\tilde{J}(R(Y,X)U)^V \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2}(\left([Y,X]^H-(R(Y,X)U)^V\right)-\frac{\sqrt{5}}{2}[Y,X]^V+(\nabla_X Y)^V) \\ &\quad +\frac{1}{2}[Y,X]^H+\frac{\sqrt{5}}{2}[Y,X]^V-\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}(R(Y,X)U)^H \\ &= -\frac{1}{2}[Y,X]^H+\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}[Y,X]^V-\frac{\sqrt{5}}{2}(\nabla_X Y)^V \\ &\quad +\frac{1}{2}[Y,X]^H+\frac{\sqrt{5}}{2}[Y,X]^V-\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}(R(Y,X)U)^H \\ &= -\frac{\sqrt{5}}{2}\left((\nabla_X Y)^V+(R(Y,X)U)^H\right) \end{aligned}$$

iii)

$$\begin{aligned}
\phi_{\tilde{J}X^V}Y^V &= -\left(L_{Y^V}\tilde{J}\right)X^V = -L_{Y^V}\tilde{J}X^V + \tilde{J}L_{Y^V}X^V \\
&= -L_{Y^V}(\frac{1}{2}X^V + \frac{\sqrt{5}}{2}X^H) \\
&= -\frac{1}{2}L_{Y^V}X^V - \frac{\sqrt{5}}{2}L_{Y^V}X^H \\
&= \frac{\sqrt{5}}{2}\left[X^H, Y^V\right] \\
&= \frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^V
\end{aligned}$$

iv)

$$\begin{aligned}
\phi_{\tilde{J}X^H}Y^V &= -\left(L_{Y^V}\tilde{J}\right)X^H = -L_{Y^V}\tilde{J}X^H + \tilde{J}L_{Y^V}X^H \\
&= -L_{Y^V}(\frac{1}{2}X^H + \frac{\sqrt{5}}{2}X^V) + \tilde{J}([Y,X]^V - (\nabla_Y X)^V) \\
&= -\frac{1}{2}L_{Y^V}X^H - \frac{\sqrt{5}}{2}L_{Y^V}X^V + \frac{1}{2}[Y,X]^V + \frac{\sqrt{5}}{2}[Y,X]^H \\
&\quad -\frac{1}{2}(\nabla_Y X)^V - \frac{\sqrt{5}}{2}(\nabla_Y X)^H \\
&= \frac{\sqrt{5}}{2}([Y,X] - (\nabla_Y X))^H \\
&= -\frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^H
\end{aligned}$$

Theorem 2.2. *Let ∇_X be the operator covariant derivation with respect to X, $\tilde{\varphi}$ be the Golden structure on $T^*(M)$ defined by (1.10) and $\phi_{\tilde{\varphi}}$ the Tachibana operator on M. We get the following formulas*

$$\begin{aligned}
i)\ \phi_{\tilde{\varphi}X^H}Y^H &= -\frac{\sqrt{5}}{2}(g(\nabla_Y X))^V + \frac{\sqrt{5}}{2}(gL_Y X)^V + \frac{\sqrt{5}}{2}\left(g^{-1}(PR(Y,X))\right)^V, \\
ii)\ \phi_{\tilde{\varphi}\omega^V}Y^H &= -\frac{\sqrt{5}}{2}\left(\left(L_Y g^{-1}\right)\circ\omega\right)^H - \frac{\sqrt{5}}{2}\left(g^{-1}(L_Y\omega)\right)^H \\
&\quad -\frac{\sqrt{5}}{2}\left(P\left(R\left(Y, g^{-1}\circ\omega\right)\right)\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}(\nabla_Y\omega)\right)^H, \\
iii)\ \phi_{\tilde{\varphi}\omega^V}\theta^V &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)}\theta)^V, \\
iv)\ \phi_{\tilde{\varphi}X^H}\omega^V &= -\frac{\sqrt{5}}{2}\left(g^{-1}(\nabla_X\omega)\right)^H,
\end{aligned}$$

where R is the curvature tensor of ∇, $\tilde{\varphi} \in \Im_1^1(M)$, $X,Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, $\tilde{X} = g\circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1}\circ\omega \in \Im_0^1(M)$.

Proof. $i)$

$$\begin{aligned}
\phi_{\tilde{\varphi}X^H}Y^H &= \left(L_{Y^H}\tilde{\varphi}\right)X^H = -L_{Y^H}\tilde{\varphi}X^H + \tilde{\varphi}L_{Y^H}X^H \\
&= -L_{Y^H}\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V + \tilde{\varphi}\left([Y,X]^H + \left(P\left(R\left(Y,X\right)\right)\right)^V\right) \\
&= -\frac{1}{2}\left[Y^H,X^H\right] - \frac{\sqrt{5}}{2}\left[Y^H,\tilde{X}^H\right] + \tilde{\varphi}([Y,X]^H + \left(P\left(R\left(Y,X\right)\right)\right)^V) \\
&= -\frac{1}{2}\left[Y,X\right]^H - \frac{1}{2}\left(PR\left(Y,X\right)\right)^V - \frac{\sqrt{5}}{2}\left(\nabla_Y\left(g\circ X\right)\right)^V + \frac{1}{2}\left[Y,X\right]^H \\
&\quad + \frac{\sqrt{5}}{2}\left[Y,X\right]^V + \frac{1}{2}\left(PR\left(Y,X\right)\right)^V + \frac{\sqrt{5}}{2}\left(PR\left(Y,X\right)\right)^V \\
&= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_Y X\right)\right)^V + \frac{\sqrt{5}}{2}\left(gL_Y X\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(PR\left(Y,X\right)\right)\right)^V
\end{aligned}$$

$ii)$

$$\begin{aligned}
\phi_{\tilde{\varphi}\omega^V}Y^H &= -\left(L_{Y^H}\tilde{\varphi}\right)\omega^V = -L_{Y^H}\tilde{\varphi}\omega^V + \tilde{\varphi}L_{Y^H}\omega^V \\
&= -L_{Y^H}\frac{1}{2}(\omega^V + \sqrt{5}\tilde{\omega}^H) + \tilde{\varphi}\left(\nabla_Y\omega\right)^V \\
&= -\frac{1}{2}L_{Y^H}\omega^V - \frac{\sqrt{5}}{2}L_{Y^H}\tilde{\omega}^H + \frac{1}{2}\left(\nabla_Y\omega\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H \\
&= -\frac{1}{2}\left(\nabla_Y\omega\right)^V - \frac{\sqrt{5}}{2}(\left[Y,g^{-1}\circ\omega\right]^H + \left(P\left(R\left(Y,g^{-1}\circ\omega\right)\right)\right)^V) \\
&\quad + \frac{1}{2}\left(\nabla_Y\omega\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(\left(L_Y g^{-1}\right)\circ\omega\right)^H - \frac{\sqrt{5}}{2}\left(g^{-1}\left(L_Y\omega\right)\right)^H \\
&\quad - \frac{\sqrt{5}}{2}\left(P\left(R\left(Y,g^{-1}\circ\omega\right)\right)\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H
\end{aligned}$$

$iii)$

$$\begin{aligned}
\phi_{\tilde{\varphi}\omega^V}\theta^V &= -\left(L_{\theta^V}\tilde{\varphi}\right)\omega^V = -L_{\theta^V}\tilde{\varphi}\omega^V + \tilde{\varphi}L_{\theta^V}\omega^V \\
&= -L_{\theta^V}\frac{1}{2}(\omega^V + \sqrt{5}\tilde{\omega}^H) \\
&= -\frac{1}{2}L_{\theta^V}\omega^V - \frac{\sqrt{5}}{2}L_{\theta^V}\tilde{\omega}^H \\
&= +\frac{\sqrt{5}}{2}\left(\nabla_{\tilde{\omega}}\theta\right)^V \\
&= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)}\theta)^V
\end{aligned}$$

iv)

$$\begin{aligned}
\phi_{\tilde{\varphi}X^H}\omega^V &= -\left(L_{\omega^V}\tilde{\varphi}\right)X^H = -L_{\omega^V}\tilde{\varphi}X^H + \tilde{\varphi}L_{\omega^V}X^H \\
&= -L_{\omega^V}\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V - \tilde{\varphi}\left(\nabla_X\omega\right)^V \\
&= -\frac{1}{2}L_{\omega^V}X^H - \frac{\sqrt{5}}{2}L_{\omega^V}\tilde{X}^V - \frac{1}{2}\left(\nabla_X\omega\right)^V \\
&\quad -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= \frac{1}{2}\left(\nabla_X\omega\right)^V - \frac{1}{2}\left(\nabla_X\omega\right)^V - \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H
\end{aligned}$$

2.2. **The Vishnevskii operators applied to vertical and horizontal lifts with respect to the Golden structure on tangent bundle and cotangent bundle.**

Definition 2.2. *Suppose now that ∇ is a linear connection on M, and let $\varphi \in \Im_1^1(M)$. We can replace the condition d) of definition 2.1 by*

$$d') \quad \psi_{\varphi X}Y = \nabla_{\varphi X}Y - \varphi\nabla_X Y \tag{2.12}$$

for any $X, Y \in \Im_0^1(M)$. Then we can consider a new operator by a Vishnevskii operator or $\psi_\varphi-$operator on M, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M) \to \Im(M)$, which satisfies conditions a), b), c), e) of definition 2.1 and the condition (d') [2, 3, 9].

Let $\omega \in \Im_1^0(M)$. Using Definition 2.2, we have

$$\begin{aligned}
\left(\psi_\varphi\omega\right)(X,Y) &= (\psi_{\varphi X}\omega)Y \\
&= (\varphi X)(\iota_Y\omega) - X(\iota_{\varphi Y}\omega) - \omega\left(\nabla_{\varphi X}Y - \varphi\left(\nabla_X Y\right)\right) \\
&= \left(\nabla_{\varphi X}\omega - \nabla_X\left(\omega\circ\varphi\right)\right)Y
\end{aligned} \tag{2.13}$$

for any $X, Y \in \Im_0^1(M)$,where $(\omega\circ\varphi)Y = \omega(\varphi Y)$. From (2.13) we see that $\psi_{\varphi X}\omega = \nabla_{\varphi X}\omega - \nabla_X(\omega\circ\varphi)$ is a $1-$form [9].

Theorem 2.3. *Let ∇^H be the horizontal lift of the Levi-Civita connection ∇ in M to $T(M)$ and $\widetilde{J}$ be the Golden structure on tangent bundle $T(M)$, which implies $\widetilde{J}^2 - \widetilde{J} - I = 0$, defined*

by (1.6). $\psi_{\tilde{J}}$ the Vishnevskii operator on M. We get the following formulas

$$
\begin{aligned}
i)\ \psi_{\tilde{J}X^V}Y^H &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H, \\
ii)\ \psi_{\tilde{J}X^V}Y^V &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V, \\
iii)\ \psi_{\tilde{J}X^H}Y^H &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V, \\
iv)\ \psi_{\tilde{J}X^H}Y^V &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H,
\end{aligned}
$$

where R is the curvature tensor of ∇, $X, Y \in \Im_0^1(M)$ and $\widetilde{J} \in \Im_1^1(M)$.

Proof. $i)$

$$
\begin{aligned}
\psi_{\tilde{J}X^V}Y^H &= \nabla^H_{\tilde{J}X^V}Y^H - \tilde{J}\nabla^H_{X^V}Y^H = \nabla^H_{\frac{1}{2}(X^V+\sqrt{5}X^H)}Y^H - \tilde{J}\nabla^H_{X^V}Y^H \\
&= \frac{1}{2}\nabla^H_{X^V}Y^H + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^H \\
&= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H
\end{aligned}
$$

$ii)$

$$
\begin{aligned}
\psi_{\tilde{J}X^V}Y^V &= \nabla^H_{\tilde{J}X^V}Y^V - \tilde{J}\nabla^H_{X^V}Y^V = \nabla^H_{\frac{1}{2}(X^V+\sqrt{5}X^H)}Y^V - \tilde{J}\nabla^H_{X^V}Y^V \\
&= \frac{1}{2}\nabla^H_{X^V}Y^V + \frac{\sqrt{5}}{2}\nabla_{X^V}Y^V \\
&= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V
\end{aligned}
$$

$iii)$

$$
\begin{aligned}
\psi_{\tilde{J}X^H}Y^H &= \nabla^H_{\tilde{J}X^H}Y^H - \tilde{J}\nabla^H_{X^H}Y^H = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V}Y^H - \tilde{J}\left(\nabla_X Y\right)^H \\
&= \frac{1}{2}\nabla^H_{X^H}Y^H + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^H - \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \\
&= \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \\
&= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V
\end{aligned}
$$

$iv)$

$$
\begin{aligned}
\psi_{\tilde{J}X^H}Y^V &= \nabla^H_{\tilde{J}X^H}Y^V - \tilde{J}\nabla^H_{X^H}Y^V = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V}Y^V - \tilde{J}\left(\nabla_X Y\right)^V \\
&= \frac{1}{2}\nabla^H_{X^H}Y^V + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^V - \frac{1}{2}\left(\nabla_X Y\right)^V - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H
\end{aligned}
$$

Theorem 2.4. *Let ∇^H be the horizontal lift of the Levi-Civita connection ∇ in M to $T^*(M)$ and $\tilde{\varphi}$ be the Golden structure on $T^*(M)$ defined by (1.10) and $\psi_{\tilde{\varphi}}$ the Vishnevskii operator on M. We get the following formulas*

$$\begin{aligned} i)\ \psi_{\tilde{\varphi}\omega^V} Y^H &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} Y)^H, \\ ii)\ \psi_{\tilde{\varphi}\omega^V} \theta^V &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} \theta)^V, \\ iii)\ \psi_{\tilde{\varphi}X^H} Y^H &= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_X Y\right)\right)^V, \\ iv)\ \psi_{\tilde{\varphi}X^H} \omega^V &= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X \omega\right)\right)^H, \end{aligned}$$

where R is the curvature tensor of ∇, $\tilde{\varphi} \in \Im_1^1(M)$, $X, Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$.

Proof. $i)$

$$\begin{aligned} \psi_{\tilde{\varphi}\omega^V} Y^H &= \nabla^H_{\tilde{\varphi}\omega^V} Y^H - \tilde{\varphi}\nabla^H_{\omega^V} Y^H = \nabla^H_{\frac{1}{2}\omega^V + \frac{\sqrt{5}}{2}(g^{-1}\circ\omega)^H} Y^H \\ &= \frac{1}{2}\nabla^H_{\omega^V} Y^H + \frac{\sqrt{5}}{2}\nabla^H_{(g^{-1}\circ\omega)^H} Y^H \\ &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} Y)^H \end{aligned}$$

$ii)$

$$\begin{aligned} \psi_{\tilde{\varphi}\omega^V} \theta^V &= \nabla^H_{\tilde{\varphi}\omega^V} \theta^V - \tilde{\varphi}\nabla^H_{\omega^V} \theta^V = \nabla^H_{\frac{1}{2}\omega^V + \frac{\sqrt{5}}{2}(g^{-1}\circ\omega)^H} \theta^V \\ &= \frac{1}{2}\nabla^H_{\omega^V} \theta^V + \frac{\sqrt{5}}{2}\nabla^H_{(g^{-1}\circ\omega)^H} \theta^V \\ &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} \theta)^V \end{aligned}$$

$iii)$

$$\begin{aligned} \psi_{\tilde{\varphi}X^H} Y^H &= \nabla^H_{\tilde{\varphi}X^H} Y^H - \tilde{\varphi}\nabla^H_{X^H} Y^H = \nabla^H_{\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V} Y^H - \tilde{\varphi}\left(\nabla_X Y\right)^H \\ &= \frac{1}{2}\left(\nabla_X Y\right)^H + \frac{\sqrt{5}}{2}\nabla^H_{(g\circ X)^V} Y^H - \frac{1}{2}\left(\nabla_X Y\right)^H \\ &\quad -\frac{\sqrt{5}}{2}\left(g\circ\left(\nabla_X Y\right)\right)^V \\ &= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_X Y\right)\right)^V \end{aligned}$$

iv)

$$
\begin{aligned}
\psi_{\tilde{\varphi}X^H}\omega^V &= \nabla^H_{\tilde{\varphi}X^H}\omega^V - \tilde{\varphi}\nabla^H_{X^H}\omega^V = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}\tilde{X}^V}\omega^V - \tilde{\varphi}\left(\nabla_X\omega\right)^V \\
&= \frac{1}{2}\left(\nabla_X\omega\right)^V + \frac{\sqrt{5}}{2}\nabla^H_{\tilde{X}^V}\omega^V - \frac{1}{2}\left(\nabla_X\omega\right)^V \\
&\quad -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H
\end{aligned}
$$

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International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:91-115

www.journalmim.com

POINTWISE SEMI-SLANT SUBMERSIONS WHOSE TOTAL MANIFOLDS ARE LOCALLY PRODUCT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study pointwise semi-slant submersions from locally product Riemannian manifolds onto Riemannian manifolds. We give example and necessary and sufficient conditions for the integrability and totally geodesicness of all distributions which are mentioned in the definition of the pointwise semi-slant submersion. Moreover, we give a characterization theorem for the proper pointwise semi-slant submersions with totally umbilical fibers and first variational formula on the fibers of a pointwise semi-slant submersion. In the view of that formula, finally we obtain necessary and sufficient condition which is new approach to check the harmonicity of a pointwise semi-slant submersion.

1. Introduction

The theory of submanifolds is a very productive area in differential geometry. In the virtue of a smooth map between Riemannian manifolds, a submersion is one of the some ways to get a submanifold. Riemannian submersions were studied first by O'Neill [19] and Gray [11]. Later Riemannian submersions considered with differentiable structures of manifolds. Watson [32] defined submersions between almost Hermitian manifolds by taking account of almost complex structure of total manifold.

Received:2018-02-17 *Revised:2018-03-15* *Accepted:2018-03-30*

2010 Mathematics Subject Classification: 53C15, 53B20.

Key words: Riemannian submersion, pointwise semi-slant submersion, horizontal distribution, locally product Riemannian manifold, first variational formula

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In this case, the vertical and horizontal distributions are invariant. Afterwards, almost Hermitian submersions have been extensively studied different subclasses of almost Hermitian manifolds, for example; see [9].

The notion of anti-invariant submersion from an almost Hermitian manifold onto a Riemannian manifold was first defined by Şahin [22]. He also studied such submersions from Kählerian manifolds onto Riemannian manifolds. In this case, the fibers are anti-invariant with respect to the almost complex structure of the total manifold of the submersion. Moreover, he studied slant [24] and semi-invariant submersions [26] under new conditions. A Lagrangian submersion [22, 28] is a special case of an anti-invariant Riemannian submersion such that the complex or almost complex structure of the total manifold reverses the vertical and horizontal distributions to each other.

Recently, it has been defined and studied that there are several new Riemannian submersions between different types of manifolds; such as slant submersions [24, 14], semi-invariant submersions [21, 26], generic submersions [2, 5], semi-slant submersions [20], pointwise slant submersions [18], anti-invariant submersions [13, 30], hemi-slant submersions [4, 29], para-contact para-complex semi-Riemannian submersions [15, 16], conformal semi-slant submersions [1], semi-slant $\xi^{\perp}-$ Riemannian submersions [3]. We note that some of these submersions have been extended to the subclasses of almost contact manifolds, for example; see [8, 27]. Recent developments on the theory of submersion could be found in the book, [23].

In the present paper, we consider pointwise semi-slant Riemannian submersions from locally product Riemannian manifolds onto Riemannian manifolds. The paper is organized as follows. In section 2, we recall the fundamental equations and notions of a Riemannian submersion. In section 3, we will provide a brief view of locally product Riemannian manifolds. We study on pointwise semi-slant submersions and give necessary and sufficient conditions for the integrability and geodesicness of the distributions which are mentioned in section 4. In particular, we give a characterization theorem for the totally umbilical fibers of the pointwise semi-slant submersions and some results for pointwise semi-slant submersions with parallel canonical structures in section 5. The last section of this paper includes a new notion. We define the first variational formula on the fibers of the pointwise semi-slant submersions. By the virtue of this formula, we prove a theorem for the harmonicity of such submersions and give some interesting results.

2. Riemannian submersions

In this section, we give necessary background for Riemannian submersions.

Let (M, g) and (N, g_N) be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective mapping $\pi : (M, g) \to (N, g_N)$ is called a *Riemannian submersion* [19] if

(S1) π has maximal rank, and

(S2) π_*, restricted to $ker\pi_*^{\perp}$, is a linear isometry.

In this case, for each $q \in N$, $\pi_q^{-1} = \pi^{-1}(q)$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $(ker\pi_*)^{\perp}$, respectively. As usual, the manifold (M, g) is called *total manifold* and the manifold (N, g_N) is called *base manifold* of the submersion $\pi : (M, g) \to (N, g_N)$. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{V}\nabla_{\mathcal{V}\bar{U}}\mathcal{H}\bar{V} + \mathcal{H}\nabla_{\mathcal{V}\bar{U}}\mathcal{V}\bar{V}, \tag{2.1}$$

$$\mathcal{A}_{\bar{U}}\bar{V} = \mathcal{V}\nabla_{\mathcal{H}\bar{U}}\mathcal{H}\bar{V} + \mathcal{H}\nabla_{\mathcal{H}\bar{U}}\mathcal{V}\bar{V} \tag{2.2}$$

for any vector fields $\bar{U}$ and $\bar{V}$ on M, where ∇ is the Levi-Civita connection of g. It is easy to see that $\mathcal{T}_{\bar{U}}$ and $\mathcal{A}_{\bar{V}}$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We now summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.4), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathrm{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if X is basic, then we have

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V. \tag{2.9}$$

Remark 2.1. *In this paper, we accept all horizontal vector fields as basic vector fields.*

It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [19] and to the book [9].

Finally, we recall that the notion of the second fundamental form of a map between Riemannian manifolds. Let (M, g) and (N, g_N) be Riemannian manifolds and $\varphi : (M, g) \to (N, g_N)$ be a smooth map. Then, the second fundamental form of φ is given by

$$(\nabla\varphi_*)(E, F) = \nabla^{\varphi}_E \varphi_* F - \varphi_*(\nabla_E F)$$

for $E, F \in \Gamma(TM)$, where ∇^{φ} is the pull back connection and we denote for convenience by ∇ the Riemannian connections of the metrics g and g_N. It is well known that the second fundamental form is symmetric [6]. Moreover, φ is said to be *totally geodesic* if $(\nabla\varphi_*)(E, F) = 0$ for all $E, F \in \Gamma(TM)$, and φ is called a *harmonic* map if $trace(\nabla\varphi_*) = 0$ [6].

3. Locally Product Riemannian manifolds

Let M be an m-dimensional manifold with a tensor field of type (1,1) such that

$$F^2 = I, (F \neq \pm I)\ , \tag{3.10}$$

where I is the identity morphism on the tangent bundle TM of M. Then we say that M is an *almost product manifold* with almost product structure F. If an almost product manifold denoted by (M, F) admits a Riemannian metric g such that

$$g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \tag{3.11}$$

for all $\bar{U}, \bar{V} \in \Gamma(TM)$, then M is called an *almost product Riemannian manifold*.

Next, we denote by ∇ the Riemannian connection with respect to g on M. We say that M is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*) if we have

$$(\nabla_{\bar{U}}\ F)\bar{V} = 0 \tag{3.12}$$

for all $\bar{U}, \bar{V} \in \Gamma(TM)$ [33].

Finally, recall that, if a manifold M can be written as a product of two totally geodesic submanifolds of it, then M is called a *locally product* of two submanifolds.

4. Pointwise Semi-Slant submersions

In this section, we will define pointwise semi-slant submersion and study on geometry of it. Before we start, we remind the definition of pointwise slant submersion.

Definition 4.1. *(*[18]*) Let π be a Riemannian submersion from an almost Hermitian manifold (M, g, J) onto a Riemannian manifold (N, g_N). If, at each given point $p \in M$, the Wirtinger angle $\theta(V)$ between JV and the space $(ker\pi_*)_p$ is independent of the choice of the nonzero vector $V \in (ker\pi_*)$, then we say that π is a* pointwise slant submersion*. In this case, the angle θ can be regarded as a function on M, which is called the* slant function *of the pointwise slant submersion.*

Now, we define a new kind of submersion as in the following.

Definition 4.2. *Let (M, g, F) be a l.p.R. manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, F) \to (N, g_N)$ is called a* pointwise semi-slant Riemannian submersion*, if there is a distribution $\mathcal{D} \subset ker\pi_*$ such that*

$$ker\pi_* = \mathcal{D} \oplus \mathcal{D}_\theta\,, \quad F\mathcal{D} = \mathcal{D}\,, \tag{4.13}$$

where $\mathcal{D}_\theta$ is orthogonal complement of $\mathcal{D}$ in $ker\pi_*$ and the angle $\theta = \theta(X)$ between FX and the space $(\mathcal{D}_\theta)_p$ is independent of the choice of nonzero vector $X \in \Gamma((\mathcal{D}_\theta)_p)$ for $p \in M$ i.e.

θ is a function on M, which is called slant function of the pointwise semi-slant submersion. We say that π is proper *if the slant function is $\theta \neq 0$ and $\theta \neq \pi/2$.*

Remark 4.1. *From now on, instead of pointwise semi-slant Riemannian submersion, we will use briefly pointwise semi-slant submersion.*

In this case, for any $V \in \Gamma(ker\pi_*)$, we have

$$V = \mathcal{P}V + \mathcal{Q}V, \tag{4.14}$$

where $\mathcal{P}V \in \Gamma(\mathcal{D})$ and $\mathcal{Q}V \in \Gamma(\mathcal{D}_\theta)$.
For $V \in \Gamma(ker\pi_*)$, we have

$$FV = \phi V + \omega V, \tag{4.15}$$

where $\phi V \in \Gamma(ker\pi_*)$ and $\omega V \in \Gamma(ker\pi_*^\perp)$.
For $\xi \in \Gamma(ker\pi_*^\perp)$, we have

$$F\xi = \mathcal{B}\xi + \mathcal{C}\xi, \tag{4.16}$$

where $\mathcal{B}\xi \in ker\pi_*$ and $\mathcal{C}\xi \in (ker\pi_*^\perp)$.
For any $E \in \Gamma(TM)$, we obtain

$$E = \mathcal{V}E + \mathcal{H}E, \tag{4.17}$$

where $\mathcal{V}E \in \Gamma(ker\pi_*)$ and $\mathcal{H}E \in \Gamma(ker\pi_*^\perp)$.
Therefore, the horizontal distribution $(ker\pi_*)^\perp$ is decomposed as

$$ker\pi_*^\perp = \omega\mathcal{D}_\theta \oplus \mu\,, \tag{4.18}$$

where μ is the orthogonal complementary distribution of $\omega\mathcal{D}_\theta$ in $(ker\pi_*^\perp)$, and it is invariant with respect to F.

Example. Consider the Euclidean 6-space $\mathbb{R}^6$ with usual metric g. Define the almost product structure F on $(\mathbb{R}^6, g)$ by

$$F\partial_1 = \partial_2, \quad F\partial_2 = \partial_1, \quad F\partial_3 = \partial_4, \quad F\partial_4 = \partial_3, \quad F\partial_5 = \partial_5, \quad F\partial_6 = -\partial_6,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \cdots, 6$ and $(x_1, x_2, \cdots, x_6)$ are natural coordinates of $\mathbb{R}^6$. Now, we define a map $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$\pi(x_1, \cdots, x_6) = \big(f_1, f_2, f_3\big),$$

where

$$f_1 = (x_1 + (\sqrt{2}-1)x_2 - x_3 + x_4 + x_6),$$
$$f_2 = (\frac{(x_1)^2}{2} + (\sqrt{2}-1)x_2 - \frac{(x_3)^2}{2} + x_4 - x_6),$$
$$f_3 = (x_1 + (\sqrt{2}-1)x_2 - x_3 - x_4 + x_6),$$

and $x_1 \neq x_3$. Then, the Jacobian matrix of π is:

$$\begin{pmatrix} 1 & \sqrt{2}-1 & -1 & 1 & 0 & 1 \\ x_1 & \sqrt{2}-1 & -x_3 & 1 & 0 & -1 \\ 1 & \sqrt{2}-1 & -1 & -1 & 0 & 1 \end{pmatrix}. \tag{4.19}$$

Since the rank of this matris is equal to 3, the map π is a submersion. After some calculations, we see that

$$ker\pi_* = \mathcal{D} \oplus \mathcal{D}_\theta \ ,$$

where

$$\mathcal{D} = span\{\partial_5\},$$

and

$$\mathcal{D}_\theta = span\{\frac{1}{\sqrt{2}}\partial_1 + \frac{1}{\sqrt{2}}\partial_2 + \partial_3, x_3\partial_1 + x_1\partial_3\}.$$

Moreover, the slant function of $\mathcal{D}_\theta$ is $\theta = \arccos(\frac{1}{2}\frac{x_3}{\sqrt{(x_1)^2+(x_3)^2}})$. By direct calculation, we see that π satisfies the condition **(S2)**. Hence the map π is a proper pointwise semi-slant submersion with the slant function θ.

Using (3.10), (4.15) and (4.16), we get the following useful facts.

Lemma 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textbf{\textit{(a)}}\ \phi^2 + \mathcal{B}\omega = I, \qquad \textbf{\textit{(b)}}\ \omega\phi + \mathcal{C}\omega = 0,$$
$$\textbf{\textit{(c)}}\ \phi\mathcal{B} + \mathcal{B}\mathcal{C} = 0, \qquad \textbf{\textit{(d)}}\ \omega\mathcal{B} + \mathcal{C}^2 = I,$$

where I is the identity operator on TM.

By using (4.13)$\sim$(4.18), we get the following two results.

Lemma 4.2. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textit{(a)}\quad \phi\mathcal{D} = \mathcal{D} \qquad \textit{(b)}\quad \phi\mathcal{D}_\theta \subset \mathcal{D}_\theta \quad \textit{(c)}\quad \omega\mathcal{D} = \{0\}.$$

Lemma 4.3. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textit{(a)}\quad \mathcal{B}(F\mathcal{D}_\theta) = \mathcal{D}_\theta \quad \textit{(b)}\quad \mathcal{B}\mu = \{0\} \quad \textit{(c)}\quad \mathcal{C}(F\mathcal{D}_\theta) = \omega\mathcal{D}_\theta \quad \textit{(d)}\quad \mathcal{C}\mu = \mu.$$

Now we investigate the effect of the almost product structure F on the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ of a pointwise semi-slant Riemannian submersion
$\pi : (M, g, F) \to (N, g_N)$.

Lemma 4.4. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W = \phi \hat{\nabla}_V W + \mathcal{B}\mathcal{T}_V W, \tag{4.20}$$

$$\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W = \omega \hat{\nabla}_V W + \mathcal{C}\mathcal{T}_V W, \tag{4.21}$$

$$\mathcal{V}\nabla_\xi \mathcal{B}\eta + \mathcal{A}_\xi \mathcal{C}\eta = \phi\mathcal{A}_\xi \eta + \mathcal{B}\mathcal{H}\nabla_\xi \eta, \tag{4.22}$$

$$\mathcal{A}_\xi \mathcal{B}\eta + \mathcal{H}\nabla_\xi \mathcal{C}\eta = \omega\mathcal{A}_\xi \eta + \mathcal{C}\mathcal{H}\nabla_\xi \eta, \tag{4.23}$$

$$\hat{\nabla}_V \mathcal{B}\xi + \mathcal{T}_V \mathcal{C}\xi = \phi\mathcal{T}_V \xi + \mathcal{B}\mathcal{H}\nabla_V \xi, \tag{4.24}$$

$$\mathcal{T}_V \mathcal{B}\xi + \mathcal{H}\nabla_V \mathcal{C}\xi = \omega\mathcal{T}_V \xi + \mathcal{C}\mathcal{H}\nabla_V \xi, \tag{4.25}$$

$$\mathcal{V}\nabla_\xi \phi V + \mathcal{A}_\xi \omega V = \mathcal{B}\mathcal{A}_\xi V + \phi\mathcal{V}\nabla_\xi V, \tag{4.26}$$

$$\mathcal{A}_\xi \phi V + \mathcal{H}\nabla_\xi \omega V = \mathcal{C}\mathcal{A}_\xi V + \omega\mathcal{V}\nabla_\xi V, \tag{4.27}$$

where $V, W \in \Gamma(ker\pi_*)$, and $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Proof. For any $V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$, using (3.12), we have

$$F\nabla_\xi V = \nabla_\xi FV.$$

Hence, using (2.7), (2.8), (4.15) and (4.16), we obtain

$$\mathcal{B}\mathcal{A}_\xi V + \mathcal{C}\mathcal{A}_\xi V + \phi\mathcal{V}\nabla_\xi V + \omega\mathcal{V}\nabla_\xi V = \mathcal{A}_\xi\phi V + \mathcal{V}\nabla_\xi\phi V + \mathcal{A}_\xi\omega V + \mathcal{H}\nabla_\xi\omega V.$$

Taking the vertical and horizontal parts of this equation, we get (4.26) and (4.27). The other assertions can be obtained by using (2.5)∼(2.8), (4.15) and (4.16).

Proposition 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we obtain*

$$\phi^2 X = \cos^2\theta X, \tag{4.28}$$

for $X \in \Gamma(\mathcal{D}_\theta)$, where θ denotes the slant function.

Proof. For any non-zero $X \in \Gamma(\mathcal{D}_\theta)$ we can write following equations:

$$\cos\theta = \frac{g(FX, \phi X)}{|FX||\phi X|} = \frac{g(X, \phi^2 X)}{|X||\phi X|} \text{ and } \cos\theta = \frac{|\phi X|}{|FX|}.$$

Then, we obtain

$$\cos^2\theta = \frac{g(X, \phi^2 X)}{|X||\phi X|}\frac{|\phi X|}{|FX|}.$$

Therefore, we get the equality

$$g(\cos^2\theta X, X) = g(X, \phi^2 X),$$

which gives the assertion.

Remark 4.2. *We easily observe that the converse of the Proposition 4.1 also holds.*

Now we give a theorem for pointwise semi-slant submersions, which has similar idea with the Theorem 4.2. in [25].

Theorem 4.1. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, π is a pointwise semi-slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

(a) $\mathcal{D}' = \{x \in \mathcal{D}' \,|\, \phi^2 X = \lambda X\}$,

(b) *For any $X \in \Gamma(TM)$, orthogonal to $\mathcal{D}'$, $\omega X = 0$.*

Moreover, in this case $\lambda = \cos^2\theta$, where θ denotes the slant function.

Proof. Let $\pi : (M, g, F) \to (N, g_N)$ be a pointwise semi-slant submersion. Then, $\lambda = \cos^2\theta$ and $\mathcal{D}' = \mathcal{D}_\theta$. By the definition of the pointwise semi-slant submersion, $\omega X = 0$, where X belongs to orthogonal complement of $\mathcal{D}'$.

Conversely, **(a)** and **(b)** imply that $TM = \mathcal{D} \oplus \mathcal{D}'$. Since $\phi\mathcal{D}' \subseteq \mathcal{D}'$, from **(b)**, $\mathcal{D}$ is an invariant distribution. Thus, π is a pointwise semi-slant submersion.

Now, we investigate the integrability conditions for invariant and slant distributions.

Theorem 4.2. *Let π be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution $\mathcal{D}$ is integrable if and only if*

$$\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in \mathcal{D}, \tag{4.29}$$

for $V, W \in \Gamma(\mathcal{D})$.

Proof. For $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$, we know $[V, W] \in \mathcal{D}$ if and only if $F[V, W] \in \mathcal{D}$. So by (4.15) we obtain,

$$\begin{aligned} g(F[V, W], X) =& g(F(\nabla_V W - \nabla_W V), X) \\ =& g(F(\mathcal{T}_V W + \hat{\nabla}_V W - \mathcal{T}_W V - \hat{\nabla}_W V), X) \\ =& g(\phi(\hat{\nabla}_V W - \hat{\nabla}_W V), X). \end{aligned}$$

Thus, $[V, W] \in \mathcal{D}$ if and only if $\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in \mathcal{D}$.

In a similar way, we get the following theorem.

Theorem 4.3. *Let π be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ is integrable if and only if*

$$\phi(\hat{\nabla}_X Y - \hat{\nabla}_Y X) \in \mathcal{D}_\theta,$$

for $X, Y \in \Gamma(\mathcal{D}_\theta)$.

If we consider the total manifold l.p.R. instead of almost product Riemannian, we obtain the following results.

Lemma 4.5. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have the followings*

$$i)\ g(\nabla_V W, X) = \csc^2\theta\{g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi W, \omega X)\} \tag{4.30}$$

$$ii)\ g(\nabla_X Y, V) = \csc^2\theta\{g(\mathcal{T}_X \omega\phi Y, V) + g(\mathcal{T}_X \omega Y, \phi V)\} \tag{4.31}$$

where θ is the slant function, $V, W \in \Gamma(\mathcal{D})$ and $X, Y \in \Gamma(\mathcal{D}_\theta)$.

Proof. Let $V, W \in \Gamma(\mathcal{D})$ and $X, Y \in \Gamma(\mathcal{D}_\theta)$. Then, by using (3.11) and (4.15), we obtain

$$\begin{aligned} g(\nabla_V W, X) =& g(\nabla_V FW, FX) \\ =& g(\nabla_V FW, \phi X) + g(\nabla_V FW, \omega X) \\ =& g(\nabla_V W, \phi^2 X) + g(\nabla_V W, \omega\phi X) + g(\nabla_V \phi W, \omega X). \end{aligned}$$

If we regard (4.28),(2.5) and (2.6) for the last expression, we get the following equality

$$(1 - \cos^2\theta) g(\nabla_V W, X) = g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi X, \omega X).$$

So, that is what we needed.

For the second equation we apply the same idea. Let $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D})$. Then by using (3.11) and (4.15), we get

$$\begin{aligned} g(\nabla_X Y, V) =& g(\nabla_X FY, FV) \\ =& g(\nabla_X \phi Y, FV) + g(\nabla_X \omega Y, FV) \\ =& g(\nabla_X \phi^2 Y, V) + g(\nabla_X \omega\phi Y, V) + g(\nabla_X \omega Y, FV). \end{aligned}$$

If we consider (4.28),(2.5) and (2.6) with the last equation, we get the following

$$\begin{aligned} g(\nabla_X Y, V) =& g(\nabla_X (\cos^2\theta) Y, V) + g(\nabla_X \omega\phi Y, V) + g(\nabla_X \omega Y, FV) \\ =& g(-(\sin 2\theta)(X\theta) Y, V) + g(\cos^2\theta \nabla_X Y, V) + g(\mathcal{T}_X \omega\phi Y, V) \\ &+ g(\mathcal{T}_X \omega Y, \phi V). \end{aligned}$$

Therefore, since $g(-(\sin 2\theta)(X\theta) Y, V) = 0$, we get the assertion.

Theorem 4.4. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution $\mathcal{D}$ is integrable if and only if*

$$g(\mathcal{T}_V \phi W - \mathcal{T}_W \phi V, \omega X) = 0,$$

for $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$.

Proof. Let $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$. Then, by Lemma 4.5 and (2.3), we have

$$\begin{aligned} g([V,W],X) =& g(\nabla_V W, X) - g(\nabla_W V, X) \\ &= \csc^2\theta\{g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi W, \omega X) \\ &- g(\mathcal{T}_W V, \omega\phi X) + g(\mathcal{T}_W \phi V, \omega X)\} \\ &= \csc^2\theta\{g(\mathcal{T}_V \phi W, \omega X) - g(\mathcal{T}_W \phi V, \omega X)\}. \end{aligned}$$

Therefore, $\mathcal{D}$ is integrable if and only if $g(\mathcal{T}_V \phi W - \mathcal{T}_W \phi V, \omega X) = 0$.

In the same way, we examine the slant distribution.

Theorem 4.5. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ is integrable if and only if*

$$g(\mathcal{T}_X \omega\phi Y - \mathcal{T}_Y \omega\phi X, V) = g(\mathcal{T}_Y \omega X - \mathcal{T}_X \omega Y, \phi V)$$

for $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma\mathcal{D}_\theta$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D})$. By using Lemma 4.5, we obtain

$$\begin{aligned} g([X,Y],V) =& \csc^2\theta\{g(\mathcal{T}_X \omega\phi Y, V) + g(\mathcal{T}_X \omega Y, \phi V) \\ &- g(\mathcal{T}_Y \omega\phi X, V) + g(\mathcal{T}_Y \omega X, \phi V)\}. \end{aligned}$$

Thus, slant distribution $\mathcal{D}_\theta$ is integrable if and only if
$g(\mathcal{T}_X \omega\phi Y - \mathcal{T}_Y \omega\phi X, V) = g(\mathcal{T}_Y \omega X - \mathcal{T}_X \omega Y, \phi V)$.

Now, we focus on that in which conditions the distributions, which we study on, define totally geodesic foliation.

Proposition 4.2. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$\mathcal{C}(\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W) + \omega(\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W) = 0, \tag{4.32}$$

for $V, W \in \Gamma(ker\pi_*)$.

Proof. For $V, W \in \Gamma(ker\pi_*)$, by using (2.5), (2.6) and (4.15), we get

$$\begin{aligned}\nabla_V W =& F\nabla_V FW = F(\nabla_V \phi W + \nabla_V \omega W)\\ =& F(\mathcal{T}_V \phi W + \hat{\nabla}_V \phi W + \mathcal{T}_V \omega W + \mathcal{H}\nabla_V \omega W)\\ =& \mathcal{B}\mathcal{T}_V \phi W + \mathcal{C}\mathcal{T}_V \phi W + \phi\hat{\nabla}_V \phi W + \omega\hat{\nabla}_V \phi W\\ &+ \phi\mathcal{T}_V \omega W + \omega\mathcal{T}_V \omega W + \mathcal{B}\mathcal{H}\nabla_V \omega W + \mathcal{C}\mathcal{H}\nabla_V \omega W.\end{aligned}$$

Therefore, $ker\pi_*$ defines a totally geodesic foliation if and only if $\mathcal{C}(\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W) + \omega(\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W) = 0$.

Proposition 4.3. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, $ker\pi_*^\perp$ defines a totally geodesic foliation if and only if*

$$\mathcal{B}(\mathcal{A}_\xi \mathcal{B}\eta + \mathcal{H}\nabla_\xi \mathcal{C}\eta) + \phi(\mathcal{V}\nabla_\xi \mathcal{B}\eta + \mathcal{A}_\xi \mathcal{C}\eta) = 0, \tag{4.33}$$

for $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Proof. This proof can likewise be done using the techniques of the proof of Proposition 4.2 .

In the view of Proposition 4.2 and Proposition 4.3, we obtain the following result.

Corollary 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, M is a locally product $M_{ker\pi_*} \times M_{ker\pi_*^\perp}$ if and only if (4.32) and (4.33) hold, where $M_{ker\pi_*}$ and $M_{ker\pi_*^\perp}$ are integral manifolds of the distributions $ker\pi_*$ and $ker\pi_*^\perp$, respectively.*

Proposition 4.4. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution $\mathcal{D}$ defines a totally geodesic foliation on $ker\pi_*$ if and only if for $U, V \in \Gamma(\mathcal{D})$,*

$$Q(\mathcal{B}\mathcal{T}_U \phi V + \phi\hat{\nabla}_U \phi V) = 0 \text{ and } (\mathcal{C}\mathcal{T}_U \phi V + \omega\hat{\nabla}_U \phi V) = 0. \tag{4.34}$$

Proof. For $U, V \in \Gamma(\mathcal{D})$, from (2.5), (2.6), (4.15) and (4.16) we obtain

$$\begin{aligned}\nabla_U V =& F\nabla_U FV = F(\nabla_U \phi V + \nabla_U \omega W)\\ =& F(\nabla_U \phi V) = F(\mathcal{T}_U \phi V + \hat{\nabla}_U \omega V)\\ =& \mathcal{B}\mathcal{T}_U \phi V + \mathcal{C}\mathcal{T}_U \phi V + \phi\hat{\nabla}_U \omega V + \omega\hat{\nabla}_U \omega V.\end{aligned}$$

Therefore, we obtain the assertion.

Proposition 4.5. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ defines a totally geodesic foliation on $ker\pi_*$ if and only if for*
$X, Y \in \Gamma(\mathcal{D}_\theta)$,

$$P(\mathcal{B}(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) + \phi(\mathcal{T}_X\omega Y + \hat{\nabla}_X\phi Y)) = 0 \tag{4.35}$$

and

$$\omega(\hat{\nabla}_X\phi Y + \mathcal{T}_X\omega Y) + \mathcal{C}(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) = 0. \tag{4.36}$$

Proof. The argument is same with the proof of Proposition 4.4.

By Proposition 4.4 and Proposition 4.5 we have the following result.

Corollary 4.2. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the vertical distribution $ker\pi_*$ is a locally product $M_\mathcal{D} \times M_{\mathcal{D}_\theta}$ if and only if (4.34) and (4.35) hold, where $M_\mathcal{D}$ and $M_{\mathcal{D}_\theta}$ are intergral manifolds of $\mathcal{D}$ and $\mathcal{D}_\theta$, respectively.*

Theorem 4.6. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, π is a totally geodesic map if and only if*

$$\omega(\hat{\nabla}_V\phi W + \mathcal{T}_U\omega W) + \mathcal{C}(\mathcal{T}_V\phi W + \mathcal{H}\nabla_V\omega W) = 0 \tag{4.37}$$

and

$$\omega(\hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi) + \mathcal{C}(\mathcal{T}_V\mathcal{B}\xi + \mathcal{H}\nabla_V\mathcal{C}\xi) = 0 \tag{4.38}$$

for $V, W \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$.

Proof. Since π is a Riemannian submersion, we have

$$(\nabla\pi_*)(\xi, \eta) = 0, \; for \; \xi, \eta \in \Gamma(ker\pi_*^\perp).$$

For $V, W \in \Gamma(ker\pi_*)$, we obtain

$$\begin{aligned}(\nabla\pi_*)(V,W) &= \nabla^{\pi}_V(\pi_*W) - \pi_*\nabla_V W \\ &= -\pi_*(F\nabla_V FW) = -\pi_*(F(\nabla_V\phi W + \nabla_V\omega W) \\ &= -\pi_*(F(\mathcal{T}_V\phi W + \hat{\nabla}_V\phi W + \mathcal{T}_V\omega W + \mathcal{H}\nabla_V\omega W) \\ &= -\pi_*(\mathcal{B}\mathcal{T}_V\phi W + \mathcal{C}\mathcal{T}_V\phi W + \phi\hat{\nabla}_V\phi W + \omega\hat{\nabla}_V\phi W \\ &+\phi\mathcal{T}_V\omega W + \omega\mathcal{T}_V\omega W + \mathcal{B}\mathcal{H}\nabla_V\omega W + \mathcal{C}\mathcal{H}\nabla_V\omega W) \\ &= -\pi_*(\mathcal{C}\mathcal{T}_V\phi W + \omega\hat{\nabla}_V\phi W + \omega\mathcal{T}_V\omega W + \mathcal{C}\mathcal{H}\nabla_V\omega W).\end{aligned}$$

Thus,

$(\nabla\pi_*)(V,W) = 0 \Leftrightarrow \omega(\hat{\nabla}_V\phi W + \mathcal{T}_V\omega W) + \mathcal{C}(\mathcal{T}_V\phi W + \mathcal{H}\nabla_V\omega W) = 0$. By a similar way above, for $V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we get

$$(\nabla\pi_*)(V,\xi) = 0 \Leftrightarrow \omega(\hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi) + \mathcal{C}(\mathcal{T}_V\mathcal{B}\xi + \mathcal{H}\nabla_V\mathcal{C}\xi) = 0.$$

Recall that the fibers of a Riemannian submersion $\pi : (M,g) \to (N,g_N)$ is called *totally umbilical* if

$$T_U V = g(U,V)H \tag{4.39}$$

for any $U, V \in \Gamma(ker\pi_*)$, where H is the mean curvature vector field of the fiber.

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) . We can define

$$(\nabla_U\phi)V = \hat{\nabla}_U\phi V - \phi\hat{\nabla}_U V, \tag{4.40}$$

$$(\nabla_U\omega)V = \mathcal{H}\nabla_U\omega V - \omega\hat{\nabla}_U V, \tag{4.41}$$

$$(\nabla_U\mathcal{B})\xi = \hat{\nabla}_U\mathcal{B}\xi - \mathcal{B}\mathcal{H}\nabla_U\xi, \tag{4.42}$$

$$(\nabla_U\mathcal{C})\xi = \mathcal{H}\nabla_U\mathcal{C}\xi - \mathcal{C}\mathcal{H}\nabla_U\xi, \tag{4.43}$$

where $U, V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

We say that ϕ (resp. ω, $\mathcal{B}$ or $\mathcal{C}$) is *parallel* if $\nabla\phi = 0$ (resp. $\nabla\omega = 0$, $\nabla\mathcal{B} = 0$ or $\nabla\mathcal{C} = 0$).

Lemma 4.6. *Let π be a pointwise semi-slant submersion with parallel canonical structures from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then for any $U, V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we have*

$$(\nabla_U \phi)V = \mathcal{B}\mathcal{T}_U V - \mathcal{T}_U \omega V, \tag{4.44}$$

$$(\nabla_U \omega)V = \mathcal{C}\mathcal{T}_U V - \mathcal{T}_U \phi V, \tag{4.45}$$

$$(\nabla_U \mathcal{B})\xi = \phi\mathcal{T}_U \xi - \mathcal{T}_U \mathcal{C}\xi, \tag{4.46}$$

$$(\nabla_U \mathcal{C})\xi = \omega\mathcal{T}_U \xi - \mathcal{T}_U \mathcal{B}\xi. \tag{4.47}$$

Proof. All of the equations follow from Lemma 4.4 and (4.40)$\sim$(4.43).

Theorem 4.7. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $dim(D_\theta) \geq 2$ and ϕ is parallel, then the fibers of π are totally geodesic or the mean curvature vector field H belongs to μ.*

Proof. If the fibers of π are totally geodesic, it is obvious. Let us assume the other case. Since $dim(D_\theta) \geq 2$, then we can choose $X, Y \in \Gamma(\mathcal{D}_\theta)$ such that the set $\{X,\ Y\}$ is orthonormal. By using (3.11), (3.12), (4.15), (4.16), (2.5) and (2.6), we have

$$\begin{aligned}
\nabla_X FY =& F\nabla_X Y \\
\nabla_X \phi Y + \nabla_X \omega Y =& F(\mathcal{T}_X Y + \hat{\nabla}_X Y) \\
\mathcal{T}_X \phi Y + \hat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y + \mathcal{H}\nabla_X \omega Y =& \mathcal{B}\mathcal{T}_X Y + \mathcal{C}\mathcal{T}_X Y + \phi\hat{\nabla}_X Y + \omega\hat{\nabla}_X Y.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
g(\hat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y, X) =& g(\mathcal{B}\mathcal{T}_X Y + \phi\hat{\nabla}_X Y, X) \\
g(\phi\hat{\nabla}_X Y - \hat{\nabla}_X \phi Y, X) =& g(\mathcal{T}_X \omega Y - \mathcal{B}\mathcal{T}_X Y, X) \\
g((\nabla_X \phi)Y, X) =& g(F\mathcal{T}_X Y - \mathcal{T}_X FY, X).
\end{aligned}$$

Since ϕ is parallel, we get

$$g(F\mathcal{T}_X Y, X) = g(\mathcal{T}_X FY, X). \tag{4.48}$$

Thus, using (4.39) and (4.48), we have

$$g(H, FY) = g(\mathcal{T}_X X, FY) = -g(\mathcal{T}_X FY, X) = -g(F\mathcal{T}_X Y, X)$$
$$= -g(\mathcal{T}_X Y, FX) = -g(X, Y)g(H, FX) = 0,$$

since $g(X, Y) = 0$. So, we deduce that $H \perp \omega\mathcal{D}_\theta$. Therefore, it follows $H \in \mu$ from (4.18).

Corollary 4.3. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $(ker\pi_*)^\perp = \omega\mathcal{D}_\theta$, i.e. $\mu = \{0\}$ and ϕ is parallel, then the fibers of π are totally geodesic.*

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we say that the fibers of π are *mixed geodesic*, if $\mathcal{T}_X W = 0$, for all $X \in \Gamma(\mathcal{D}_\theta)$, $W \in \Gamma(\mathcal{D})$, [26].

Theorem 4.8. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If ω is parallel, i.e. $\nabla\omega = 0$, then the fibers of π are mixed geodesic.*

Proof. Let ω be parallel, then for any $U, V \in \Gamma(ker\pi_*)$ from (4.45), we have

$$\mathcal{C}\mathcal{T}_U V = \mathcal{T}_U \phi V. \tag{4.49}$$

Using (4.49), we obtain

$$\mathcal{C}^2\mathcal{T}_U V = \mathcal{T}_U \phi^2 V. \tag{4.50}$$

If we put $U = W \in \Gamma(\mathcal{D})$ and $V = X \in \Gamma(\mathcal{D}_\theta)$ in (4.50) and using (4.28), we get

$$\mathcal{C}^2\mathcal{T}_W X = \cos^2\theta\mathcal{T}_W X. \tag{4.51}$$

On the other hand, using the symmetry property of $\mathcal{T}$ on $\Gamma(ker\pi_*)$ and (4.49), we have

$$\mathcal{C}^2\mathcal{T}_W X = \mathcal{C}^2\mathcal{T}_X W = \mathcal{T}_X \phi^2 W = \mathcal{T}_X W, \tag{4.52}$$

that is

$$\mathcal{C}^2\mathcal{T}_W X = \mathcal{T}_X W. \tag{4.53}$$

Since submersion π is proper, from (4.51) and (4.53), it follows that

$$\mathcal{T}_X W = 0. \tag{4.54}$$

Remark 4.3. *Most of our results for pointwise semi-slant submersion is similar to semi-slant case, see* [12].

5. The first variational form of a pointwise semi-slant submersion

In this section, we give a different approach to check whether a submersion is harmonic and define the first variational form of a semi-slant submersion.

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we can define the 1-form dual to the vector field $F\xi$, for $\xi \in \Gamma(ker\pi_*^{\perp})$, such that

$$\sigma_\xi : \Gamma(ker\pi_*) \mapsto \mathcal{F}(\pi_q^{-1}), q \in N$$

$$V \mapsto \sigma_\xi(V) = g(F\xi, V),$$

for all $V \in \Gamma(ker\pi_*)$. In the view of [31] and [7], we define the followings.

The *Legendre variations* of any fiber of π, denoted by the set $\mathbb{L}$, where

$$\mathbb{L} = \{\xi \in \Gamma(ker\pi_*^{\perp}) : d\sigma_\xi = 0, \ i.e. \ \sigma_\xi \ is \ closed\},$$

the *Hamiltonian variations* of any fiber of π, denoted by the set $\mathbb{E}$,

$$\mathbb{E} = \{\xi \in \Gamma(ker\pi_*)^{\perp} : \exists f \in \mathcal{F}(\pi_q^{-1}) \Rightarrow \sigma_\xi = df, \ i.e. \ \sigma_\xi \ is \ exact\}$$

and the *harmonic variations* of any fiber of π are given by the set

$$\mathbb{H} = \{\xi \in \Gamma(ker\pi_*)^{\perp} : \Delta\sigma_\xi = 0; \ i.e. \ \sigma_\xi \ is \ harmonic\}.$$

By the definitions of differential and co-differential operators, we observe that

$$\mathbb{E} \subset \mathbb{L}\,, \ \ \mathbb{H} \subset \mathbb{L} \ \ and \ \ \mathbb{E} \cap \mathbb{H} = 0. \tag{5.55}$$

Now, we examine that in which conditions the 1-form σ_ξ defined above is a Legendre variation.

Lemma 5.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). The 1-form σ_ξ is a Legendre variation if and only if*

$$g(\mathcal{T}_U\xi, \phi V) - g(\mathcal{T}_V\xi, \phi U) = g(\mathcal{A}_\xi U, \omega V) - g(\mathcal{A}_\xi V, \omega U), \tag{5.56}$$

for all $U, V \in \Gamma(ker\pi_*)$.

Proof. Let U, V be in $ker\pi_*$. Then, by the definition of differential, (2.6) and (3.11), we obtain

$$\begin{aligned}(d\sigma_\xi)(U,V) =& Ug(F\xi,V) - Vg(F\xi,U) - g(F\xi,[U,V]) \\ =& Ug(\xi,FV) - Vg(\xi,FU) - g(\xi,F[U,V]) \\ =& g(\nabla_U\xi,FV) + g(\xi,\nabla_U FV) \\ & - g(\nabla_V\xi,FU) - g(\xi,\nabla_V FU) \\ & - g(\xi,F\nabla_U V) + g(\xi,F\nabla_V U) \\ =& g(\nabla_U\xi,\phi V + \omega V) - g(\nabla_V\xi,\phi U + \omega U) \\ =& g(\nabla_U\xi,\phi V) + g(\nabla_U\xi,\omega V) \\ & - g(\nabla_V\xi,\phi U) + g(\nabla_V\xi,\omega U) \\ =& g(\mathcal{T}_U\xi,\phi V) + g(\mathcal{H}\nabla_U\xi,\omega V) \\ & - g(\mathcal{T}_V\xi,\phi U) + g(\mathcal{H}\nabla_V\xi,\omega U).\end{aligned}$$

Since we may assume ξ is basic, we have

$$\begin{aligned}(d\sigma_\xi)(U,V) =& g(\mathcal{T}_U\xi,\phi V) + g(\mathcal{A}_\xi U,\omega V) \\ & - g(\mathcal{T}_V\xi,\phi U) + g(\mathcal{A}_\xi V,\omega U).\end{aligned}$$

Thus, the assertion follows.

Lemma 5.2. *For $\xi \in \Gamma(\mu)$, $\sigma_\xi \equiv 0$.*

Proof. Let $\xi \in \Gamma(\mu)$. Then, $F\xi \in \Gamma(\mu)$. For any $V \in \Gamma(ker\pi_*)$, we get

$$\sigma_\xi(V) = g(F\xi,V) = 0.$$

So, $\sigma_\xi \equiv 0$, for all $V \in \Gamma(ker\pi_*)$.

Remark 5.1. *Because of Lemma 5.2, throughout this paper, we can assume that H belongs to $\Gamma(\omega\mathcal{D}_\theta)$.*

Proposition 5.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) and f be a smooth function on a fiber. Then, $F(grad(f)|_{\omega\mathcal{D}_\theta}) \in \mathbb{E}$.*

Proof. Let f be a smooth function on a fiber. For $\xi = F(grad(f)|_{\omega\mathcal{D}_\theta})$, and any $V \in \Gamma(ker\pi_*)$, we obtain

$$\sigma_\xi(V) = g(F\xi, V) = g(grad(f), V) = V[f] = df(V).$$

Thus, we get $\sigma_\xi = df$, i.e. $\xi \in \mathbb{E}$.

Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N) and $\xi \in \Gamma(ker\pi_*^\perp)$. The first variation of the volume form of a fiber π_q^{-1}, for $q \in N$, is defined as follows [17]

$$\mathbf{V}'(\xi) = -k \int_{\pi_q^{-1}} g(\xi, H)*1. \tag{5.57}$$

where $k = dim(\pi_q^{-1})$. We call the fibers;

- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{L}$, then π_q^{-1} is $\mathbb{L} - minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{E}$, then π_q^{-1} is $\mathbb{E} - minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{H}$, then π_q^{-1} is $\mathbb{H} - minimal$.

Remark 5.2. *One can easily see that if the fiber is minimal, then the fiber is $\mathbb{L}, \mathbb{E}$ and $\mathbb{H} - minimal$. On the other hand, because of the facts that $\mathbb{E} \subset \mathbb{L}$ and $\mathbb{H} \subset \mathbb{L}$, the fiber is $\mathbb{E} - minimal$ and $\mathbb{H} - minimal$ if it is $\mathbb{L} - minimal$.*

Theorem 5.1. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*

(a) *The fiber π_q^{-1} is $\mathbb{L} - minimal$ if and only if σ_H is co-exact.*
(b) *The fiber π_q^{-1} is $\mathbb{E} - minimal$ if and only if σ_H is co-closed.*
(c) *The fiber π_q^{-1} is $\mathbb{H} - minimal$ if and only if σ_H is the sum of an exact and a co-exact 1-form.*

Proof.

(a) $\Rightarrow$: Let the fiber π_q^{-1} is $\mathbb{L} - minimal$, then for any $\xi \in \mathbb{L}$, we have $g(H, \xi) = 0$ from (5.57). By the definition of the Hodge star operator [10], we have

$$\sigma_\xi \wedge \sigma_H(V_1, V_2, ..., V_k) = g(\xi, H)*1(V_1, V_2, ..., V_k),$$

for $V_1, V_2, ..., V_k \in \Gamma(ker\pi_*)$. From the definition of the global scalar product $(.|.)$ (see [10]) on the module of all forms on the fiber, we get

$$(\sigma_\xi|\sigma_H) = \int_{\pi_q^{-1}} \sigma_\xi \wedge *\sigma_H = 0. \tag{5.58}$$

Denote by δ the codifferential operator [10] on the fiber π_q^{-1}. Since σ_ξ is closed, for any 2-form β on π_q^{-1}, we have

$$0 = (d\sigma_\xi|\beta) = (\sigma_\xi|\delta\beta). \tag{5.59}$$

Since π_q^{-1} is compact, by (5.58) and (5.59) we conclude that σ_H is co-exact.
$\Leftarrow:$ Suppose that σ_H is co-exact, we have $\sigma_H = \delta\psi$ for some 2-form ψ. Then, for any $\xi \in \mathbb{L}$,

$$(\sigma_\xi|\sigma_H) = (\sigma_\xi|\delta\psi) = (d\sigma_\xi|\psi) = 0$$

and then

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi^{-1}(q)} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0,$$

i.e. π_q^{-1} is $\mathbb{L} - minimal$.
(b) $\Rightarrow:$ Let the fiber π_q^{-1} be $\mathbb{E} - minimal$. Then, we have

$$0 = \mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, $(\sigma_\xi|\sigma_H) = 0$. Since $\xi \in \mathbb{E}$, $\sigma_\xi = df$ for some function f on the fiber π_q^{-1}. Thus,

$$(df|\sigma_H) = (f|\delta\sigma_H) = 0.$$

Hence it follows that $\delta\sigma_H = 0$, i.e. σ_H is co-closed.
$\Leftarrow:$ Suppose that σ_H is co-closed. Let $\xi \in \mathbb{E}$, then there exists a function $f \in \mathcal{F}(\pi_q^{-1})$ such that $\sigma_\xi = df$. Hence, we have

$$(\sigma_\xi|\sigma_H) = (df|\sigma_H) = (f|\delta\sigma_H) = 0.$$

Therefore,

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0,$$

that is $\mathbf{V}^{'}(\xi) = 0$ for $\xi \in \mathbb{E}$, i.e. π_q^{-1} is $\mathbb{E} - minimal$.
(c) $\Rightarrow:$ If the fiber π_q^{-1} is $\mathbb{H} - minimal$, then for $\xi \in \mathbb{H}$, we have

$$0 = \mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H).$$

It means that, σ_H is orthogonal to harmonic 1-forms on the fiber π_q^{-1}. Thus, by the Hodge decomposition theorem [10], we conclude that σ_H is the sum of an exact and a co-exact 1-form.

$\Leftarrow:$ Let σ_H be the sum of an exact 1-form $\omega_1 = df$ and a co-exact 1-form $\omega_2 = \delta\psi$. For $\xi \in \mathbb{H}$, we have

$$\begin{aligned}(\sigma_\xi|\sigma_H) =&(\sigma_\xi|df + \delta\psi) = (\sigma_\xi|df) + (\sigma_\xi|\delta\psi)\\ =&(\delta\sigma_\xi|f) + (d\sigma_\xi|\psi) = 0,\end{aligned}$$

since $d\sigma_\xi = \delta\sigma_\xi = 0$. Thus,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi, H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, the fiber is $\mathbb{H} - minimal$.

Theorem 5.2. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $H \in \mathbb{L}$, then*

(a) *π_q^{-1} is $\mathbb{L} - minimal$ if and only if π_q^{-1} is minimal.*
(b) *π_q^{-1} is $\mathbb{E} - minimal$ if and only if σ_H is a harmonic variation.*
(c) *π_q^{-1} is $\mathbb{H} - minimal$ if and only if σ_H is an exact 1-form.*

Proof. **(a)** If the fiber π_q^{-1} is $\mathbb{L} - minimal$, then by Theorem 5.1-(a) we have, σ_H is co-exact. Hence σ_H is co-closed. Taking into account the fact that $d\sigma_H = 0$, we deduce that σ_H is harmonic. But this is a contradiction because of Hodge decomposition theorem [10]. So, σ_H must be zero. Hence we conclude that $H = 0$. The converse is clear.

(b) $\Rightarrow:$ If the fiber π_q^{-1} is $\mathbb{E} - minimal$, then we have $\delta\sigma_H = 0$ from Theorem 5.1-(b). Since $d\sigma_H = 0$, σ_H is also harmonic, i.e. $\Delta\sigma_H = 0$.

$\Leftarrow:$ If σ_H is harmonic, then σ_H is co-closed. By Theorem 5.1-(b), the fiber π_q^{-1} is $\mathbb{E} - minimal$.

(c) $\Rightarrow:$ Assume that π_q^{-1} is $\mathbb{H} - minimal$. Then, from Theorem 5.1-(c), σ_H is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $H \in \mathbb{L}$ implies that σ_H is orthogonal to every co-exact 1-form on π_q^{-1}. Thus, σ_H must be exact.

$\Leftarrow$: Let σ_H be an exact 1-form. For $\xi \in \mathbb{H}$, we obtain

$$\begin{aligned}\mathbf{V}^{'}(\xi) =& - k\int_{\pi_q^{-1}} g(\xi, H) {*1} = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H)\\ =& - k(\sigma_\xi|\sigma_H) = (\sigma_\xi|df) = (\delta\sigma_\xi|f) = 0,\end{aligned}$$

that is, π_q^{-1} is $\mathbb{H} - minimal$.

Remark 5.3. *It is well known that, the fibers of a submerion is minimal if and only if the submersion is harmonic. Now, we give a new approach for harmonicity of a pointwise semi-slant submersion. By Theorem 5.2-(a), we obtain the following result.*

Corollary 5.1. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $H \in \mathbb{L}$, then π is harmonic if and only if π_q^{-1} is $\mathbb{L} - minimal$.*

Lemma 5.3. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\mathcal{T}_{\phi E_i} E_i, H) = \Sigma_i g(\mathcal{A}_{\omega E_i} E_i, H), \tag{5.60}$$

where $\{E_1, E_2, ..., E_m\}$ is a local basis of $\mathcal{D}_\theta$.

Proof.

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i} FH, E_i) = 0.$$

Using (3.12),

$$\begin{aligned}\Rightarrow \delta\sigma_H = 0 \Leftrightarrow & \Sigma_i g(\nabla_{E_i} H, FE_i) \Leftrightarrow \Sigma_i g(\nabla_{E_i} H, \phi E_i + \omega E_i)\\ =& \Sigma_i g(\nabla_{E_i} H, \phi E_i) + \Sigma_i g(\nabla_{E_i} H, \omega E_i)\\ =& \Sigma_i g(\mathcal{T}_{E_i} H, \phi E_i) + \Sigma_i g(\mathcal{A}_H E_i, \omega E_i).\end{aligned}$$

Thus, the assertion follows from the skew-symmetry and symmetry properties of the O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$.

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International Journal of Maps in Mathematics
Volume:1, Issue:1, (2018), Pages:116-136
www.journalmim.com

ON f-BIHARMONIC HYPERSURFACES

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Abstract. In the present paper, we study spacelike and timelike f-biharmonic hypersurfaces in Lorentzian para-Sasakian manifolds. We investigate f-biharmonic equations for spacelike hypersurfaces of an Lorentzian para-Sasakian manifold with a constant and harmonic mean curvature. Also we give some results for f-biharmonic timelike hypersurfaces in η-Einstein Lorentzian para-Sasakian manifolds.

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1. Introduction

Harmonic maps have an important area of study as a generalization of important ideas like geodesics and minimal submanifolds. A significant literature has been created in the last decade including relationships between harmonic maps and the other disciplines namely theoretical physics.

A map $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds is called harmonic if it is a critical point of energy functional given by

$$E(\Psi) = \frac{1}{2}\int_{\Omega} \mid d\Psi \mid^2 \vartheta_g.$$

Therefore harmonic maps are the solutions of the corresponding Euler-Lagrange equation,

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Received:2018-03-13 *Accepted:2018-04-09*

2010 Mathematics Subject Classification: 53C42, 53C25, 53D15.

Key words: Biharmonic map, f-biharmonic map, Lorentzian para-Sasakian manifold.

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which is characterized by the vanishing of the tension field

$$\tau(\Psi) = trace\nabla d\Psi.$$

As introduced by J. Eells and J. H. Sampson in [1], bienergy of a map Ψ is defined by

$$E_2(\Psi) = \frac{1}{2}\int_{\Omega} \mid \tau(\Psi) \mid^2 \vartheta_g,$$

and Ψ is said to be biharmonic if it is a critical point of the bienergy.

In [2], the first and second variation formula for the bienergy were derived by G. Y. Jiang, showing that the Euler-Lagrange equation associated to E_2 is

$$\tau_2(\Psi) = -\triangle\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where $\Delta = -trace(\nabla^{\Psi}\nabla^{\Psi} - \nabla^{\Psi}_{\nabla})$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator of N. The equation $\tau_2(\Psi) = 0$ is called biharmonic equation and it is clear that any harmonic map is biharmonic.

With another aspect, B. Y. Chen [5] defined biharmonic submanifolds of the Euclidean space by $\Delta H = 0$, that is any submanifold in Euclidean space whose mean curvature vector field H is harmonic is called a biharmonic submanifold, where Δ is the Laplacian of the submanifold acting on functions. Also B. Y. Chen [5] made a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, that is minimal. If one use the definition of biharmonic maps to Riemannian immersions into Euclidean space, it is easy to see that Chen's definition of biharmonic submanifold coincides with the definition given by using bienergy functional.

In the literature there are many results on the non-existence of biharmonic submanifolds in manifolds with non-positive sectional curvature. These non-existence consequences (see [3, 6],) and as well as Generalized Chen's conjecture: any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal, which was proposed by R. Caddeo, S. Montaldo and C. Oniciuc [7], led the studies to spheres and other non-negatively curved spaces. But in recent years the authors of [8] proved that generalized Chen conjecture is not true by constructing examples of proper biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature.

In the last fifteen years there is a growing interest in biharmonic maps theory and its applications to the other areas. For some recent geometric studies of general biharmonic maps and biharmonic submanifolds see [9, 7, 10, 8, 11, 15, 12, 13, 14] and the references therein.

f-harmonic maps between Riemannian manifolds were first introduced and studied by A. Lichnerowicz in 1970 (see also [16]). A smooth map $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds is called f-harmonic if it is a critical point of f-energy functional defined by

$$E_f(\Psi) = \frac{1}{2}\int_\Omega f \mid d\Psi \mid^2 \vartheta_g,$$

for any compact domain domain Ω. The Euler-Lagrange equation is given by

$$\tau_f(\Psi) = f\tau(\Psi) + d\Psi(gradf) = 0,$$

where $\tau(\Psi)$ is the tension field of Ψ. If f is a constant function then it is obvious that f-harmonic maps are harmonic. So f-harmonic maps, where f is a non-constant function are more interesting to study. We call such maps as proper f-harmonic maps.

The concept of f-biharmonic maps have been introduced by W.-J. Lu [17] as a generalization of biharmonic maps. A differentiable map between Riemannian manifolds is said to be f-biharmonic if it is a critical point of the f-bienergy functional defined by integral of f times the square-norm of the tension field, where f is a smooth positive function on the domain. If $f = 1$ then f-biharmonic maps are biharmonic. To avoid the confussion with the types of map called by the same name in [18] and defined as a critical point of the square-norm of the f-tension field, some authors (see [17], [19]) called the map defined in [18] as bi-f-harmonic map.

In the present paper, our aim is to study f-biharmonic equations for hypersurfaces in Lorentzian para-Sasakian manifolds. Since the characteristic vector field of a Lorentzian para-Sasakian manifold is timelike then we consider it in the tangent space and as the normal vector of the hypersurface, respectively. We investigate f-biharmonic equations for spacelike hypersurfaces of an LP-Sasakian manifold with a constant and harmonic mean curvature. Also we give some results for f-biharmonic timelike hypersurfaces in η-Einstein LP-Sasakian manifolds.

2. Preliminaries

2.1. **Harmonic maps.** A map $\Psi \in C^\infty(M, N)$ is called *harmonic* if it is a critical point of the *energy* functional

$$E(\Psi) = \frac{1}{2}\int_\Omega |d\Psi|^2 v_g, \tag{2.1}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the harmonic map equation [1]

$$\tau(\Psi) \equiv trace \nabla d\Psi = 0, \tag{2.2}$$

where $\tau(\Psi) \equiv trace\nabla d\Psi$ is called the tension field of Ψ, ∇ is a connection induced from the Levi-Civita connection ∇^M of M and the pull-back connection ∇^Ψ.

2.2. **Biharmonic maps.** As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J. H. Sampson in [1]. *Biharmonic maps* between Riemannian manifolds $\Psi : (M, g) \to (N, h)$ are the critical points of the *bienergy functional*

$$E_2(\Psi) = \frac{1}{2}\int_\Omega |\tau(\Psi)|^2 v_g. \tag{2.3}$$

The first variation formula for the bienergy which is derived in [2, 3] shows that the Euler-Lagrange equation for the bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\Delta\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0, \tag{2.4}$$

where $\Delta = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of $\Psi^{-1}TN$ and $R^N(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator on N. From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps which are called proper biharmonic maps are more interesting.

2.3. ***f*-Harmonic maps.** f-Harmonic maps are critical points of the f-energy functional for maps $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_f(\Psi) = \frac{1}{2}\int_\Omega f\, |d\Psi|^2 v_g, \tag{2.5}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the f-harmonic map equation ([20], [18]):

$$\tau_f\,(\Psi) \equiv f\,\tau(\Psi) + d\Psi(gradf) = 0, \tag{2.6}$$

where $\tau(\Psi) \equiv trace\nabla d\Psi$ is the tension field of Ψ vanishing of which means Ψ is a harmonic map. $\tau_f\,(\Psi)$ is called the f-tension field of map Ψ.

2.4. **f-Biharmonic maps.** f-Biharmonic maps are critical points of the f-bienergy functional for maps $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E_f(\Psi) = \frac{1}{2}\int_{\Omega} f \; |\tau(\Psi)|^2 v_g, \tag{2.7}$$

where Ω is a compact domain of M. The Euler-Lagrange equation gives the f-biharmonic map equation [17] :

$$\tau_{2,f}(\Psi) \equiv f\,\tau_2(\Psi) + (\Delta f)\tau(\Psi) + 2\nabla^{\Psi}_{gradf}\,\tau(\Psi) = 0, \tag{2.8}$$

where $\tau(\Psi)$ and $\tau_2(\Psi)$ are the tension and bitension fields of Ψ, respectively. $\tau_{2,f}(\Psi)$ called the f-bitension field of map Ψ.

2.5. **Bi-f-Harmonic maps.** Bi-f-harmonic maps are critical points of the bi-f-energy functional for maps $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds:

$$E^2_f(\Psi) = \frac{1}{2}\int_{\Omega} |\tau_f(\Psi)|^2 v_g. \tag{2.9}$$

The Euler-Lagrange equation gives the bi-f-harmonic map equation [18] :

$$\tau^2_f(\Psi) \equiv f\,J^{\Psi}(\tau_f(\Psi)) - \nabla^{\Psi}_{gradf}\,\tau_f(\Psi), \tag{2.10}$$

where J^{Ψ} is the Jacobi operator of the map defined by

$$J^{\Psi}(X) = -Tr\,(\nabla^{\Psi}\nabla^{\Psi}X - \nabla^{\Psi}_{\nabla^M}X - R^N(d\Psi, X)d\Psi). \tag{2.11}$$

The following illustrate the relations among these different types of harmonic maps:

$$\text{Harmonic maps} \subset \text{Biharmonic maps} \subset f\text{-Biharmonic maps},$$

$$\text{Harmonic maps} \subset f\text{-Harmonic maps} \subset \text{Bi-}f\text{-harmonic maps}.$$

3. Lorentzian Para-Sasakian Manifolds

Let $\overline{M}$ be an $(m+1)$-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\overline{M}$ such that [21]

$$\eta(\xi) = -1, \tag{2.2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2.2}$$

where I denotes the identity map of $T_p\overline{M}$ and $\otimes$ is the tensor product. The equations (2.2.1) and (2.2.2) imply that

$$\begin{aligned} \eta \circ \phi &= 0, \\ \phi\xi &= 0, \\ rank(\phi) &= m. \end{aligned} \tag{2.2.3}$$

Then $\overline{M}$ admits a Lorentzian metric $\overline{g}$, i.e., $\overline{g}$ is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate inner product of index 1, where $T_p\overline{M}$ is the tangent space of $\overline{M}$ at the point p,such that

$$\overline{g}(\phi X, \phi Y) = \overline{g}(X,Y) + \eta(X)\eta(Y), \tag{2.2.4}$$

and $\overline{M}$ is said to admit a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$. Then we get

$$\begin{aligned} \overline{g}(X,\xi) &= \eta(X), \\ \Phi(X,Y) &= \overline{g}(X,\phi Y) = \overline{g}(\phi X, Y) = \Phi(Y,X), \\ (\overline{\nabla}_X \Phi)(Y,Z) &= g(Y, (\overline{\nabla}_X \phi)Z) = (\overline{\nabla}_X \Phi)(Z,Y), \end{aligned} \tag{2.2.5}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to $\overline{g}$. A non-zero vector $X_p \in T_p\overline{M}$ is called spacelike, null or timelike ,if it satisfies $\overline{g}_p(X_p, X_p) \geq 0$, $\overline{g}_p(X_p, X_p) = 0$ $(X_p \neq 0)$ or $\overline{g}_p(X_p, X_p) < 0$, respectively. It is clear that the Lorentzian metric $\overline{g}$ makes ξ a timelike unit vector field, i.e, $\overline{g}(\xi,\xi) = -1$. The manifold $\overline{M}$ equipped with a Lorentzian almost paracontact structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian almost paracontact manifold (for short LAP-manifold) [21], [22].

In equations (2.2.1) and (2.2.2) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on M defined by Satō [23].

A Lorentzian almost paracontact manifold $\overline{M}$ endowed with the structure $(\phi, \xi, \eta, \overline{g})$ is called a Lorentzian para-Sasakian manifold (for short LP-Sasakian) [21] if

$$(\overline{\nabla}_X \phi)Y = \eta(Y)X + \overline{g}(X,Y)\xi + 2\eta(X)\eta(Y)\xi. \tag{2.2.6}$$

In an LP-Sasakian manifold the 1-form η is closed and

$$\overline{\nabla}_X \xi = \phi X. \tag{2.2.7}$$

Also, an LP-Sasakian manifold $\overline{M}$ is said to be η-Einstein if its Ricci tensor $\overline{S}$ satisfies

$$\overline{S}(X,Y) = a\overline{g}(X,Y) + b\eta(X)\eta(Y), \tag{2.2.8}$$

for any vector fields X, Y, where a and b are functions on $\overline{M}$. The Ricci tensor of an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold is given by [24]

$$\overline{S}(X,Y) = (\frac{\overline{r}}{m} - 1)\overline{g}(X,Y) + (\frac{\overline{r}}{m} - (m+1))\eta(X)\eta(Y), \tag{2.2.9}$$

where $\overline{r}$ is the scalar curvature of the manifold.

In an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$ with the structure $(\phi, \xi, \eta, \overline{g})$, the following relations hold [21], [25]:

$$\overline{g}(\overline{R}(X,Y)Z,\xi) = \eta(\overline{R}(X,Y)Z) = \overline{g}(Y,Z)\eta(X) - \overline{g}(X,Z)\eta(Y), \tag{2.2.10}$$

$$\overline{R}(\xi,X)Y = \overline{g}(X,Y)\xi - \eta(Y)X \tag{2.2.11}$$

$$\overline{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.2.12}$$

$$\overline{R}(\xi,X)\xi = X + \eta(X)\xi, \tag{2.2.13}$$

$$\overline{S}(X,\xi) = m\eta(X), \tag{2.2.14}$$

$$\overline{S}(\phi X,\phi Y) = \overline{S}(X,Y) + m\eta(X)\eta(Y), \tag{2.2.15}$$

for any vector fields X, Y, Z in $\overline{M}$, where $\overline{R}$ and $\overline{S}$ are the Riemannian curvature and the Ricci tensors of $\overline{M}$, respectively.

A semi-Riemannian hypersurface of a semi-Riemannian manifold is just a semi-Riemannian submanifold of codimension 1. It is well known that a Lorentzian manifold is a semi-Riemannian manifold with a symmetric nondegenerate $(0,2)$ tensor field, namely metric tensor, of index 1. Let M be a hypersurface of a Lorentzian manifold $\overline{M}$. If the normal vector field of M is timelike(respectively, spacelike) then M is called a spacelike (respectively, timelike) hypersurface of $\overline{M}$ (see [26]).

Let M be a hypersurface of an $(m+1)$-dimensional LP-Sasakian manifold $\overline{M}$. The Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{2.2.16}$$

$$\overline{\nabla}_X N = -A_N X, \tag{2.2.17}$$

for each X, $Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where ∇ is the Levi-Civita connection on M, B is the second fundamental form of M and A_N is the shape operator with respect to the normal section N. We can write $B(X,Y) = b(X,Y)N$, where b is the function-valued second

fundamental form of M. Then the second fundamental form B and the shape operator of the hypersurface with respect to the unit normal vector field N are related by

$$B(X,Y)=\varepsilon\overline{g}(\overline{\nabla}_X Y,N)N=-\varepsilon\overline{g}(Y,\overline{\nabla}_X N)N=\varepsilon\overline{g}(A_N X,Y)N \tag{2.2.18}$$

and

$$\overline{g}(A_N X,Y)=\overline{g}(B(X,Y),N)=\overline{g}(b(X,Y)N,N)=\varepsilon b(X,Y), \tag{2.2.19}$$

where X, $Y\in\Gamma(TM)$, $N\in\Gamma(T^{\perp}M)$ and $\varepsilon=\overline{g}(N,N)$.

4. f-Biharmonic spacelike hypersurfaces in LP-Sasakian manifolds

In this section we consider that the characteristic vector field of the LP-Sasakian manifold is the unit normal vector field of the hypersurface. Hence, we characterize the spacelike f-biharmonic hypersurfaces in a Lorentzian para-Sasakian (LP-Sasakian) manifold.

Definition 4.1. *A hypersurface in an LP-Sasakian manifold is called an f-biharmonic hypersurface if the isometric immersion defining the hypersurface is an f-biharmonic map.*

Minimal hypersurfaces are well known examples of biharmonic hypersurfaces. Also biharmonic hypersurfaces are f-biharmonic with the function $f=1$. So we have the following relationship:

Minimal hypersurfaces $\subset$ Biharmonic hypersurfaces $\subset$ f-Biharmonic hypersurfaces.

Neither minimal nor biharmonic hypersurfaces will be called proper f-biharmonic submanifolds.

The f-biharmonic equation for a hypersurface in a Riemannian manifold is given in the following [19].

Theorem 4.1. *Let $\Psi: M^m\to N^{m+1}$ be an isometric immersion of codimension-one with mean curvature vector $\eta=H\xi$. Then Ψ is an f-biharmonic map if and only if*

$$\begin{cases}\Delta H-H\,|A|^2+HRic^N(\xi,\xi)+H\frac{\Delta f}{f}+2(grad\ln f)H=0,\\ 2A(gradH)+\frac{m}{2}gradH^2-2H(Ric^N(\xi))^T+2HA(grad\ln f)=0,\end{cases} \tag{3.1}$$

where $Ric^N: T_qN\to T_qN$ denotes the Ricci operator of the ambient space, A is the shape operator of the hypersurface with respect to the unit normal vector ξ, and Δ, $grad$ are the Laplace and the gradient operator of the hypersurface, respectively.

Theorem 4.2. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and $\Psi : M \to \overline{M}$ be an isometric immersion with $\dim M = m$. Assume that the characteristic vector field ξ is the unit normal vector field of the hypersurface M. Then the spacelike hypersurface M is f-biharmonic if and only if*

$$\begin{aligned} \frac{m}{2} f\,(gradH^2) - 2A(gradfH) &= 0, & (3.2)\\ -\Delta(fH) + 2H\Delta f + 2mfH &= 0 & (3.3) \end{aligned}$$

where A is the shape operator of the hypersurface with respect to the unit normal vector field ξ and $\mu = H\xi$ is the mean curvature vector.

Proof. Let M be a hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field ξ and $\Psi : M \to \overline{M}$ be an isometric immersion. Assume that $\{e_i\}_{i=1}^m$ is a local orthonormal frame of M such that $\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$ is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$. We identify $d\Psi(X)$ by X and $\nabla^{\Psi}_X W$ by $\overline{\nabla}_X W$ for all $X \in \Gamma(TM)$, $W \in \Gamma(\Psi^{-1}TM)$. Note that the tension field of Ψ is $\tau(\Psi) = mH\xi$. Then the bitension field of $\Psi : M \to \overline{M}$ is as follows:

$$\begin{aligned} \tau_2(\psi) &= \sum_{i=1}^m \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}\tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}\tau(\Psi) - \overline{R}(d\Psi(e_i), \tau(\Psi))d\Psi(e_i)\} \\ &= \sum_{i=1}^m \{\nabla^{\Psi}_{e_i}\nabla^{\Psi}_{e_i}(mH\xi) - \nabla^{\Psi}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\} \\ &= \sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(mH\xi) - \overline{\nabla}_{\nabla_{e_i}e_i}(mH\xi) - \overline{R}(d\Psi(e_i), mH\xi)d\Psi(e_i)\} \\ &= m\sum_{i=1}^m \{\overline{\nabla}_{e_i}\left(e_i(H)\xi + H\overline{\nabla}_{e_i}\xi\right) - \left(\nabla_{e_i}e_i\right)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi \\ &\quad - H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\} \\ &= m\sum_{i=1}^m \{e_ie_i(H)\xi + 2e_i(H)\overline{\nabla}_{e_i}\xi + H\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi \\ &\quad - \left(\nabla_{e_i}e_i\right)(H)\xi - H\overline{\nabla}_{\nabla_{e_i}e_i}\xi - H\overline{R}(d\Psi(e_i), \xi)d\Psi(e_i)\} \\ &= -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + mH\sum_{i=1}^m \overline{R}(\xi, d\Psi(e_i))d\Psi(e_i). \quad (3.4) \end{aligned}$$

Since $\overline{M}$ is a LP-Sasakian manifold then from (2.2.11), we have

$$\sum_{i=1}^m \overline{R}(\xi, d\Psi(e_i))d\Psi(e_i) = m\xi. \quad (3.5)$$

By writing (3.5) in (3.4), we get

$$\tau_2(\Psi) = -m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + m^2 H\xi. \tag{3.6}$$

From (2.8) we can write f-bitension field of Ψ :

$$\begin{aligned} \tau_{2,f}(\Psi) \equiv \; & f\left\{-m(\Delta H)\xi - mH\Delta^{\Psi}\xi - 2mA(gradH) + m^2 H\xi\right\} \\ & + m(\Delta f)H\xi + 2m\nabla^{\Psi}_{gradf}(H\xi) \end{aligned}$$

Now, to compute the tangential and normal parts of the f-bitenson field, it suffices to find only the normal and tangential parts of $\Delta^{\Psi}\xi$ and $\nabla^{\Psi}_{gradf}(H\xi)$:

From (2.2.7) we have

$$\begin{aligned} \overline{g}(\Delta^{\Psi}\xi,\xi) &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi,\xi) \\ &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi,\xi) \\ &= \sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi,\overline{\nabla}_{e_i}\xi), \\ &= \sum_{i=1}^{m}\overline{g}(\phi e_i,\phi e_i). \end{aligned} \tag{3.7}$$

By using (2.2.4) in (3.7), then the normal part of $\Delta^{\Psi}\xi$ is

$$\begin{aligned} \left(\Delta^{\Psi}\xi\right)^{\perp} &= -\overline{g}(\Delta^{\Psi}\xi,\xi)\xi \\ &= -\sum_{i=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\xi,\overline{\nabla}_{e_i}\xi)\xi \\ &= -m\xi. \end{aligned} \tag{3.8}$$

The tangential part of $\Delta^{\Psi}\xi$ can be calculated by

$$\begin{aligned}
\left(\Delta^{\Psi}\xi\right)^{\top} &= -\sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi, e_k)e_k \\
&= \sum_{i,k=1}^{m} \overline{g}(\overline{\nabla}_{e_i}Ae_i - A(\nabla_{e_i}e_i), e_k)e_k \\
&= \sum_{i,k=1}^{m} \{e_i\overline{g}(Ae_i, e_k) - \overline{g}(Ae_i, \nabla_{e_i}e_k) - \overline{g}(A(\nabla_{e_i}e_i), e_k)\}e_k \\
&= \sum_{i,k=1}^{m} \{-e_i b(e_i, e_k) + b(e_i, \nabla_{e_i}e_k) + b(\nabla_{e_i}e_i, e_k)\}e_k \\
&= -\sum_{i,k=1}^{m} \{\nabla_{e_i}b(e_k, e_i)\}e_k. \qquad (3.9)
\end{aligned}$$

By Codazzi-Mainardi equation, we have

$$\begin{aligned}
\sum_{i=1}^{m}(\nabla_{e_i}b(e_k, e_i) - \nabla_{e_k}b(e_i, e_i)) &= -\sum_{i=1}^{m}\overline{g}\left(\overline{R}(e_i, e_k)e_i, \xi\right) \\
&= \overline{S}(\xi, e_k). \qquad (3.10)
\end{aligned}$$

Since $\overline{S}(\xi, e_k) = 0$, (3.10) implies that

$$\sum_{i=1}^{m}(\nabla_{e_i}b(e_k, e_i) - \nabla_{e_k}b(e_i, e_i)) = 0. \qquad (3.11)$$

If we write (3.11) in (3.9), we get

$$\left(\Delta^{\Psi}\xi\right)^{\top} = -mgradH. \qquad (3.12)$$

On the other hand we have

$$\begin{aligned}
\nabla^{\Psi}_{gradf}\ (H\xi) &= \overline{\nabla}_{gradf}(H\xi) \\
&= gradf\,(H)\xi + H\overline{\nabla}_{gradf}\,\xi \\
&= g(gradf, gradH)\xi - HA(gradf), \qquad (3.13)
\end{aligned}$$

which implies

$$\nabla^{\Psi}_{gradf}\ (H\xi) = div(fgradH)\xi - (f\,\Delta H)\xi - HA(gradf). \qquad (3.14)$$

Then the tangential and normal parts of $\nabla^{\Psi}_{gradf}\ (H\xi)$ are given by

$$\begin{aligned}
\left[\nabla^{\Psi}_{gradf}\ (H\xi)\right]^{\top} &= -HA(gradf), \qquad (3.15) \\
\left[\nabla^{\Psi}_{gradf}\ (H\xi)\right]^{\perp} &= div(fgradH)\xi - (f\,\Delta H)\xi, \qquad (3.16)
\end{aligned}$$

respectively. Finally by considering all these parts, we have the tangential and normal components of the bitension field as follows:

$$(\tau_{2,f}\,(\Psi))^{\top} = -2mA(gradfH) + \frac{m^2}{2} f\,(gradH^2), \tag{3.17}$$

$$(\tau_{2,f}\,(\Psi))^{\perp} = -m\Delta(fH) + 2m^2 fH + 2mH\Delta f \tag{3.18}$$

This completes the proof.

Corollary 4.1. *A spacelike hypersurface of an LP-Sasakian manifold with a costant mean curvature is f-biharmonic if and only if either it is minimal or*

$$\begin{cases} A(gradf) = 0, \\ \Delta f = -2mf. \end{cases} \tag{3.19}$$

Proof. Let M be a spacelike hypersurface of an $(m+1)$-dimensional LP-Sasakian manifold with H =constant. From (3.2) and (3.3) we have M is an f-biharmonic hypersurface if and only if

$$\begin{aligned} HA(gradf) &= 0, \\ H\Delta f + 2mfH &= 0, \end{aligned}$$

by virtue of Weitzenböck formula. This completes the proof.

Corollary 4.2. *A spacelike hypersurface of an LP-Sasakian manifold with a harmonic mean curvature is f-biharmonic if and only if*

$$\begin{cases} 2A(grad(\ln fH) = mgradH, \\ g(grad\ln f, grad\ln H) = \frac{\Delta f}{2f} + m. \end{cases} \tag{3.20}$$

From Corollary 4.1 it is obvious that f-biharmonic spacelike hypersurfaces of LP-Sasakian manifolds with a constant mean curvature are either minimal ones or satisfy (3.19).

Theorem 4.3. *Let M be a totally umbilic f-biharmonic spacelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with dimension $(m+1)$. Then we have*

$$\begin{cases} \left(\frac{m}{2}+1\right) grad\lambda = A(grad\ln f), \\ 2g(gradf, gradH) = f\,\Delta\lambda - \lambda\,(\Delta f + 2mf)\,. \end{cases} \tag{3.21}$$

Proof. Assume that $\{e_i\}_{i=1}^{m}$ is a local orthonormal frame of M such that

$$\{d\Psi(e_1), d\Psi(e_2), ..., d\Psi(e_m), \xi\}$$

is an adapted orthonormal frame of the LP-Sasakian manifold $\overline{M}$ where $\Psi : M \to \overline{M}$ is an isometric immersion. By identifying $d\Psi(X)$ by X, for all X in TM, we have an orthonormal basis$\{e_1, e_2, ..., e_m, \xi\}$ for the ambient manifold $\overline{M}$ such that $Ae_i = \lambda_i e_i$, where A is the shape operator of M and λ_i, $(1 \leq i \leq m)$, is the principal curvatures in the direction of e_i. Since M is totally umbilical then all the principal curvatures at any point p of M are equal to the same number $\lambda(p)$. Then by taking ξ instead of N in (2.2.18) we have

$$\begin{aligned} H &= -\frac{1}{m}\sum_{i=1}^{m} g(B(e_i, e_i), \xi) \\ &= -\frac{1}{m}\sum_{i=1}^{m} g(Ae_i, e_i) \\ &= -\frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i, e_i) \\ &= -\lambda. \end{aligned} \tag{3.22}$$

On the other hand by using (3.22), we get

$$A(gradH) = -\frac{1}{2}grad\lambda^2. \tag{3.23}$$

Since M is an f-biharmonic spacelike hypersurface of $\overline{M}$, from (3.2), (3.3), (3.22), (3.23) and Weitzenböck formula we complete the proof.

Corollary 4.3. *Let M be a totally umbilic f-biharmonic spacelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with dimension $(m+1)$. If $gradf \perp gradH$ then we have*

$$\begin{cases} \left(\frac{m}{2}+1\right) f\, gradH = -A(grad\, f), \\ \frac{\Delta H}{H} = \frac{\Delta f}{f} + 2m. \end{cases} \tag{3.24}$$

5. f-Biharmonic timelike hypersurfaces in LP-Sasakian manifolds

Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be a hypersurface of $\overline{M}$. Assume that the characteristic vector field of $\overline{M}$ belongs to the tangent hyperplane of the hypersurface M and N is the unit normal vector field of the manifold. Since N is spacelike then M becomes timelike hypersurface of $\overline{M}$.

We note that the tension field of the isometric immersion $\Psi : M \to \overline{M}$ is

$$\tau(\Psi) = m\mu,$$

where $\mu = HN$ is the mean curvature vector field with the mean curvature function H.

Theorem 5.1. *Let $(\overline{M}, \phi, \xi, \eta, \overline{g})$ be an $(m+1)$-dimensional LP-Sasakian manifold and M be its timelike hypersurface. Then M is an f-biharmonic hypersurface of $\overline{M}$ if and only if*

$$\begin{cases} \frac{m}{2} f\, gradH^2 + 2A(grad f H) - 2f\, H(\overline{Q}(N)) = 0, \\ f\, H\, |A|^2 + \Delta\,(f\, H) - 2H\,\,(\Delta f) - f\, H\, \overline{S}(N,N) = 0. \end{cases} \tag{4.1}$$

where $\overline{S}$ is the Ricci curvature of the LP-Sasakian manifold $\overline{M}$, $\overline{Q}$ is the Ricci operator of $\overline{M}$ defined by $\overline{g}(\overline{Q}X, Y) = \overline{S}(X, Y)$ and A is the shape operator of the hypersurface with respect to the unit normal vector field N.

Proof. Assume that M is a timelike hypersurface of the LP-Sasakian manifold $\overline{M}$ with the unit normal vector field N and $\Psi : M \to \overline{M}$ be an isometric immersion. Consider $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an local orthonormal basis for the hypersurface. Since the tension field of Ψ is $\tau(\Psi) = mHN$, we have

$$\begin{aligned}
\tau_2(\psi) &= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i} \nabla^{\Psi}_{e_i} \tau(\Psi) - \nabla^{\Psi}_{\nabla_{e_i} e_i} \tau(\Psi) - \overline{R}(d\Psi(e_i), \tau(\Psi)) d\Psi(e_i)\} \\
&= \sum_{i=1}^{m} \varepsilon_i \{\nabla^{\Psi}_{e_i} \nabla^{\Psi}_{e_i} (mHN) - \nabla^{\Psi}_{\nabla_{e_i} e_i} (mHN) - \overline{R}(d\Psi(e_i), mHN) d\Psi(e_i)\} \\
&= \sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} (mHN) - \overline{\nabla}_{\nabla_{e_i} e_i} (mHN) - \overline{R}(d\Psi(e_i), mHN) d\Psi(e_i)\} \\
&= m \sum_{i=1}^{m} \varepsilon_i \{\overline{\nabla}_{e_i} \left(e_i(H) N + H \overline{\nabla}_{e_i} N\right) - (\nabla_{e_i} e_i)(H) N - H \overline{\nabla}_{\nabla_{e_i} e_i} N \\
&\quad - H \overline{R}(d\Psi(e_i), N) d\Psi(e_i)\} \\
&= m \sum_{i=1}^{m} \varepsilon_i \{e_i e_i(H) N + 2 e_i(H) \overline{\nabla}_{e_i} N + H \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N \\
&\quad - (\nabla_{e_i} e_i)(H) N - H \overline{\nabla}_{\nabla_{e_i} e_i} N - H \overline{R}(d\Psi(e_i), N) d\Psi(e_i)\} \\
&= -m(\Delta H) N - mH \Delta^{\Psi} N - 2mA(gradH) \\
&\quad - mH \left\{ \sum_{i=1}^{m-1} \overline{R}(d\Psi(e_i), N) d\Psi(e_i) - \overline{R}(d\Psi(\xi), N) d\Psi(\xi) \right\},
\end{aligned} \tag{4.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$, $\overline{R}$ is the Riemannian curvature tensor of $\overline{M}$ and ∇^{Ψ} is the pull-back connection.

Now we shall compute the tangential and normal components of the $\Delta^{\Psi} N$ and the curvature term, respectively:

The tangential part of $\Delta^{\Psi}N$ can be calculated by

$$\begin{aligned}
\left(\Delta^{\Psi}N\right)^{\top} &= -\sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N-\overline{\nabla}_{\nabla_{e_i}e_i}N,e_k)e_k \\
&= \sum_{i,k=1}^{m}\overline{g}(\overline{\nabla}_{e_i}Ae_i-A(\nabla_{e_i}e_i),e_k)e_k \\
&= \sum_{i,k=1}^{m}\{e_i\overline{g}(Ae_i,e_k)-\overline{g}(Ae_i,\nabla_{e_i}e_k)-\overline{g}(A(\nabla_{e_i}e_i),e_k)\}e_k \\
&= \sum_{i,k=1}^{m}\{e_ib(e_i,e_k)-b(e_i,\nabla_{e_i}e_k)-b(\nabla_{e_i}e_i,e_k)\}e_k \\
&= \sum_{i,k=1}^{m}\{\nabla_{e_i}b(e_k,e_i)\}e_k, \qquad (4.3)
\end{aligned}$$

where ∇ is the induced connection of the hypersurface and $b:\Gamma(TM)\times\Gamma(TM)\to C^{\infty}(M,R)$ is the function valued second fundamental form such that $B(X,Y)=b(X,Y)N$, for all vector fields X, Y on M. By Codazzi-Mainardi equation, we have

$$\begin{aligned}
\sum_{i=1}^{m}(\nabla_{e_i}b(e_k,e_i)-\nabla_{e_k}b(e_i,e_i)) &= \sum_{i=1}^{m}\overline{g}\left(\overline{R}(e_i,e_k)e_i,N\right) \\
&= -\overline{S}(N,e_k). \qquad (4.4)
\end{aligned}$$

which implies that

$$\sum_{i=1}^{m}\nabla_{e_i}b(e_k,e_i)=\sum_{i=1}^{m}\nabla_{e_k}b(e_i,e_i)-\overline{S}(N,e_k). \qquad (4.5)$$

By using (4.5) in (4.3), we obtain

$$\begin{aligned}
\left(\Delta^{\Psi}N\right)^{\top} &= \sum_{i,k=1}^{m}\{\nabla_{e_k}b(e_i,e_i)-\overline{S}(N,e_k)\}e_k \\
&= m(gradH)-(\overline{Q}(N)). \qquad (4.6)
\end{aligned}$$

By straightforward computations, the normal part of the $\Delta^{\Psi}N$ is

$$\begin{aligned}
(\Delta^{\Psi}N)^{\perp} &= \overline{g}(\Delta^{\Psi}N,N)N \\
&= -\sum_{i=1}^{m}\{\varepsilon_i\overline{g}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N-\overline{\nabla}_{\nabla_{e_i}e_i}N,N)\}N \\
&= \sum_{i=1}^{m}\{\varepsilon_i\overline{g}(\overline{\nabla}_{e_i}N,\overline{\nabla}_{e_i}N)\}N \\
&= |A|^2 N, \qquad (4.7)
\end{aligned}$$

where $\varepsilon_i=\overline{g}(e_i,e_i)$, $1\leq i\leq m$.

On the other hand, since

$$\begin{aligned} -\sum_{k=1}^{m-1}\overline{S}(N,e_k)e_k &= \sum_{i,k=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),e_k)e_k \\ &\quad -\sum_{i=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),\xi)\xi \\ &= (\overline{Q}(N))\xi \qquad (4.8) \end{aligned}$$

and

$$\sum_{i=1}^{m-1}\overline{g}(\overline{R}(d\Psi(e_i),N)d\Psi(e_i),N) = -\overline{S}(N,N)+1 \qquad (4.9)$$

then the tangential and normal components of the first curvature term in (4.2) are equal to $(\overline{Q}(N))$ and $-\overline{S}(N,N)+1$, respectively. Also, from (2.2.13) we have

$$\overline{R}(d\Psi(\xi),N)d\Psi(\xi) = N. \qquad (4.10)$$

Hence we get

$$\begin{aligned} (\tau_2(\psi))^{\top} &= -m\left[\frac{m}{2}(gradH^2)+2A(gradH)-2H(\overline{Q}(N))\right], \qquad (4.11) \\ (\tau_2(\psi))^{\perp} &= -m\left[(\Delta H)+H\,|A|^2-H\overline{S}(N,N)\right]N. \qquad (4.12) \end{aligned}$$

Also we have

$$\begin{aligned} \nabla^{\Psi}_{gradf}\,(HN) &= \overline{\nabla}_{gradf}(HN) \\ &= gradf\,(H)N+H\overline{\nabla}_{gradf}\,N \\ &= g(gradf,gradH)N-HA(gradf), \qquad (4.13) \end{aligned}$$

which implies

$$\nabla^{\Psi}_{gradf}\,(HN) = div(fgradH)N-(f\,\Delta H)N-HA(gradf). \qquad (4.14)$$

Then the tangential and normal parts of $\nabla^{\Psi}_{gradf}\,(HN)$ are given by

$$\begin{aligned} \left[\nabla^{\Psi}_{gradf}\,(HN)\right]^{\top} &= -HA(gradf), \qquad (4.15) \\ \left[\nabla^{\Psi}_{gradf}\,(HN)\right]^{\perp} &= div(fgradH)N-(f\,\Delta H)N, \qquad (4.16) \end{aligned}$$

By reorganizing all the tangent and normal parts of the f-bitension field, we get

$$
\begin{aligned}
(\tau_{2,f}(\psi))^{\top} &= f\left\{-m\left(\frac{m}{2}(gradH^2)+2A(gradH)-2H(\overline{Q}(N))\right)\right\}-2mHA(gradf)\\
&= -\frac{m^2}{2}f\,gradH^2-2mfA(gradH)+2mf\,H(\overline{Q}(N))-2mHA(gradf)\\
&= -m\left\{\frac{m}{2}f\,gradH^2+2A(gradfH)-2f\,H(\overline{Q}(N))\right\} \qquad (4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
(\tau_{2,f}(\psi))^{\perp} &= f\left\{-m\left[(\Delta H)+H\,|A|^2-H\overline{S}(N,N)\right]N\right\}\\
&\quad +m\,(\Delta f)\,HN+2m\,\{g(gradf,gradH)N\}\\
&= -mf\,(\Delta H)N-mf\,H\,|A|^2\,N+mf\,H\,\overline{S}(N,N)N \qquad (4.18)\\
&\quad +m\,(\Delta f)\,HN+2mdiv(Hgradf)N-2mH\,(\Delta f)\,N\\
&= -m\left\{f\,H\,|A|^2+\Delta\,(f\,H)-2H\,(\Delta f)-f\,H\,\overline{S}(N,N)\right\}N,
\end{aligned}
$$

which give

$$
\begin{aligned}
\tau_{2,f}(\psi) &= -m\left\{\frac{m}{2}f\,gradH^2+2A(gradfH)-2f\,H(\overline{Q}(N))\right\} \qquad (4.19)\\
&\quad -m\left\{f\,H\,|A|^2+\Delta\,(f\,H)-2H\,(\Delta f)-f\,H\,\overline{S}(N,N)\right\}N.
\end{aligned}
$$

This completes the proof.

Corollary 5.1. *Let M be a timelike hypersurface of an LP-Sasakian manifold $\overline{M}$ with constant mean curvature. Then M is an f-biharmonic timelike hypersurface if and only if either it is minimal or*

$$
\begin{cases}
A(gradf)=f\,(\overline{Q}(N)),\\
\frac{\Delta f}{f}=|A|^2-\overline{S}(N,N).
\end{cases} \qquad (4.20)
$$

Corollary 5.2. *A timelike hypersurface of a Ricci flat LP-Sasakian manifold with a constant mean curvature is f-biharmonic if and only if*

$$
\begin{cases}
A(gradf)=0,\\
\frac{\Delta f}{f}=|A|^2.
\end{cases} \qquad (4.21)
$$

Theorem 5.2. *Let $\overline{M}$ be an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold and M be a timelike hypersurface of $\overline{M}$. Then M is f-biharmonic if and only if*

$$
\begin{cases}
A(grad\ln(fH))=-\frac{m}{2}\,(gradH)\,,\\
\frac{\Delta(f\,H)}{f\,H}-2\frac{(\Delta f)}{f}=-\,|A|^2+\frac{\overline{r}}{m}-1,
\end{cases} \qquad (4.22)
$$

where $\overline{r}$ is the scalar curvature of $\overline{M}$. Particularly, if M is a timelike hypersurface with $0 \neq H =$constant, then M is a non-minimal f-biharmonic timelike hypersurface if and only if

$$\begin{cases} A(gradf) = 0, \\ \frac{\Delta f}{f} = |A|^2 - \frac{\overline{r}}{m} + 1. \end{cases} \tag{4.23}$$

Proof. Assume that $\overline{M}$ be an $(m+1)$-dimensional η-Einstein LP-Sasakian manifold. Then by using (2.2.9), we have

$$\overline{S}(N,N) = \frac{\overline{r}}{m} - 1. \tag{4.24}$$

On the other hand

$$(\overline{Q}(N)) = 0. \tag{4.25}$$

By using (4.24) and (4.25) in (4.1), we obtain the assertion of the theorem.

Theorem 5.3. *Let $\overline{M}$ be an $(m+1)$-dimensional LP-Sasakian space form and M be a timelike hypersurface of $\overline{M}$. Then M is f-biharmonic if and only if*

$$\begin{cases} A(grad \ln(fH)) = -\frac{m}{2}\left(gradH\right), \\ \frac{\Delta(f\,H)}{f\,H} - 2\frac{\Delta f}{f} = -\left|A\right|^2 + m, \end{cases} \tag{4.26}$$

In particular, M is a hypersurface of $\overline{M}$ with a constant mean curvature, then M is a non-minimal biharmonic timelike hypersurface if and only if $\left|A\right|^2 = \frac{\Delta f}{f} + m$.

Proof. In an $(m+1)$-dimensional LP-Sasakian space form $\overline{M}$, since

$$\overline{S}(X,Y) = m\overline{g}(X,Y),$$

for all vector fields X, Y , then $\overline{M}$ is an η-Einstein manifold with

$$\overline{r} = m(m+1). \tag{4.27}$$

Therefore, the f-biharmonic equation reduces to (4.26).

Theorem 5.4. *Let $\overline{M}$ be an $(m+1)$-dimensional (dim $\overline{M} > 2$) η-Einstein LP-Sasakian manifold. If M is a totally umbilical f-biharmonic timelike hypersurface of $\overline{M}$ then we have*

$$\begin{aligned} A(gradf) &= 2fgrad\lambda, \\ \frac{\Delta\left(f\,\lambda\right)}{f\,\lambda} - 2\frac{(\Delta f)}{f} &= -m\lambda^2 + \frac{\overline{r}}{m} - 1. \end{aligned}$$

Proof. Let $\{e_1, e_2, ..., e_{m-1}, e_m = \xi, N\}$ be a local orthonarmal basis of η-Einstein LP-Sasakian manifold $\overline{M}$ such that $\{e_1, e_2, ..., e_{m-1}, e_m = \xi\}$ is an orthonormal frame for the hypersurface M. Since M is totally umbilical, we have $A = \lambda I$, where λ is a smooth function. Then

$$\begin{aligned} H &= \frac{1}{m}\sum_{i=1}^{m}\varepsilon_i \overline{g}(B(e_i, e_i), N) \\ &= \frac{1}{m}\sum_{i=1}^{m-1}\overline{g}(Ae_i, e_i) \\ &= \frac{1}{m}\sum_{i=1}^{m} g(\lambda e_i, e_i) \\ &= \frac{m-1}{m}\lambda. \end{aligned} \tag{4.28}$$

From (4.28), we can write

$$A(gradH) = \frac{m-1}{2m} grad\lambda^2. \tag{4.29}$$

On the other hand, by straightforward calculations one can easily see that

$$|A|^2 = m\lambda^2. \tag{4.30}$$

By using (4.28), (4.29) and (4.30) in (4.22), we obtain

$$\begin{aligned} fA(gradH) + HA(gradf) &= -\frac{m}{4} f\left(gradH^2\right) \\ f\frac{m-1}{2m} grad\lambda^2 + \frac{m-1}{m}\lambda A(gradf) &= -\frac{m}{2} f\left(\frac{m-1}{m}\right)^2 grad\lambda^2 \\ \lambda A(gradf) &= f\left[\frac{1}{2} - m\right] grad\lambda^2 \\ A(gradf) &= 2fgrad\lambda \end{aligned}$$

and

$$\begin{aligned} \frac{\Delta(fH)}{fH} - 2\frac{(\Delta f)}{f} &= -|A|^2 + \frac{\overline{r}}{m} - 1, \\ \frac{\Delta(f\lambda)}{f\lambda} - 2\frac{(\Delta f)}{f} &= -m\lambda^2 + \frac{\overline{r}}{m} - 1, \end{aligned}$$

which complete the proof.

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